Gazale formulas, a new class of the hyperbolic Fibonacci and Lucas functions, and the improved method of the "golden" cryptography

Формулы Газале, новый класс гиперболических функций Фибоначчи и Люка и усовершенствованный метод «золотой» криптографии

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Расширенная аннотация

В конце 20-го века сразу несколько исследователей из разных стран (Вера Шпинадель, Джей Капрафф, Мидхат Газале, Александр Татаренко и др.) независимо друг от друга обратили внимание на следующее обобщение рекуррентного соотношения Фибоначчи: $F_m(n+2) = mF_m(n+1) + F_m(n)$, которое, в свою очередь, приводит к следующему обобщению уравнения золотой пропорции: $x^2 - mx - 1 = 0$, где m – положительное действительное число, названное Мидхатом Газале *«гномонным» числом* или *порядковым числом рекуррентного соотношения Фибоначчи.* Будем называть положительный корень введенного выше квадратного уравнения *обобщенным золотым сечением порядка m* $\Phi_m = \frac{\sqrt{4+m^2}+m}{2}$. В своей книге «Гномон. От фараонов до фракталов» [9], опубликованной в 1999 г. и переведенной на русский язык в 2002 г., Газале вывел следующую замечательную формулу, которая задает аналитически обобщенные числа Фибоначчи $F_m(n)$ в диапазоне значений $n = 0, \pm1, \pm2, \pm3, ...$:

$$F_m(n) = \frac{\Phi_m^n - (-1)^n \Phi_m^{-n}}{\sqrt{4+m^2}}$$

Следует отметить, что выведенная формула задает бесконечное количество новых рекуррентных последовательностей, подобных числам Фибоначчи, так как каждому m соответствует своя числовая последовательность. Некоторые из них приведены в таблице ниже:

Обобщенные числа Фибоначчи порядка *m*=1, 2, 3, 4

m	Φ_m	-5	-4	-3	-2	-1	0	1	2	3	4	5
1	$\frac{1+\sqrt{5}}{2}$	5	-3	2	-1	1	0	1	1	2	3	5
2	$1+\sqrt{2}$	29	-12	5	-2	1	0	1	2	5	12	29
3	$\frac{3+\sqrt{13}}{2}$	109	-33	10	-3	1	0	1	3	10	33	109
4	$2+\sqrt{5}$	305	-72	17	-4	1	0	1	4	17	72	305

Заметим, что второй ряд этой таблицы (m=1) задает классические числа Фибоначчи, в то время как третий ряд (m=2) задает еще один замечательный числовой ряд, известный под названием *числа Пелли.*

Эта формула по праву может быть отнесена к разряду выдающихся математических формул наряду с формулами Эйлера, формулами Муавра, формулами Бине и т.д. Автор настоящей статьи предлагает назвать эту формулу *формулой Газале.*

Именно *формула Газале* вдохновила автора на получение следующих новых математических результатов:

(1) Выведена следующая формула:

$$L_m(n) = \Phi_m^n + (-1)^n \Phi_m^{-n} .$$

Эта формулу автор назвал *формулой Газале для обобщенных чисел Люка порядка m,* поскольку автор выполнил лишь техническую работу при выводе данной формула. Заметим, что эта формула задает бесконечное количество новых рекуррентных последовательностей, частными случаями которых являются классические *числа Люка* (m=1) и *числа Пелли-Люка* (m=2). Некоторые из этих числовых последовательностей приведены в таблице ниже:

Обобщенные числа Люка порядка *m*=1, 2, 3, 4

m	Φ_m	-5	-4	-3	-2	-1	0	1	2	3	4	5
1	$\frac{1+\sqrt{5}}{2}$	-11	7	-4	3	-1	2	1	3	4	7	11
2	$1+\sqrt{2}$	-82	34	-14	6	-2	2	2	6	14	34	82
3	$\frac{3+\sqrt{13}}{2}$	-393	119	-36	11	-3	2	3	11	36	119	393
4	$2+\sqrt{5}$	-1364	322	-76	18	-4	2	4	18	76	322	1364

(2) Следующим научным результатом является введение нового класса гиперболических функций Фибоначчи и Люка, основанных на формулах Газале:

<u>Гиперболический синус Фибоначчи порядка m</u>

$$sF_m(x) = \frac{\Phi_m^x - \Phi_m^{-x}}{\sqrt{4+m^2}} = \frac{1}{\sqrt{4+m^2}}\left[\left(\frac{m+\sqrt{4+m^2}}{2}\right)^x - \left(\frac{m+\sqrt{4+m^2}}{2}\right)^{-x}\right]$$

<u>Гиперболический косинус Фибоначчи порядка m</u>

$$cF_m(x) = \frac{\Phi_m^x + \Phi_m^{-x}}{\sqrt{4+m^2}} = \frac{1}{\sqrt{4+m^2}}\left[\left(\frac{m+\sqrt{4+m^2}}{2}\right)^x + \left(\frac{m+\sqrt{4+m^2}}{2}\right)^{-x}\right]$$

<u>Гиперболический синус Люка порядка m</u>

$$sL_m(x) = \Phi_m^x - \Phi_m^{-x} = \left(\frac{m+\sqrt{4+m^2}}{2}\right)^x - \left(\frac{m+\sqrt{4+\sqrt{4+m^2}}}{2}\right)^{-x}$$

<u>Гиперболический косинус Люка порядка m</u>

$$cL_m(x) = \Phi_m^x + \Phi_m^{-x} = \left(\frac{m+\sqrt{4+m^2}}{2}\right)^x + \left(\frac{m+\sqrt{4+\sqrt{4+m^2}}}{2}\right)^{-x}$$

Заметим, что эти гиперболические функции являются обобщением *симметричных гиперболических функций Фибоначчи и Люка*, введенными Стаховым и Розиным в 2005 г. [14].

Трудно вообразить, что количество новых гиперболических функций Фибоначчи и Люка бесконечно, так как каждому m (m – положительное действительное число) соответствует свой вариант гиперболических функций. Новый класс гиперболических функций представляет собой фундаментальный интерес для гиперболической геометрии и теоретической физики и может привести к переосмысливанию «гиперболической геометрии Лобачевского» и «пространства Минковского» (гиперболической интерпретации специальной теории относительности

Эйнштейна).

(3) В развитие Q-матицы $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, исследованной американским математиком Вернером Хоггаттом, создателем Фибоначчи-Ассоциации, в настоящей работе введено понятие Q_m-матрицы $G_m = \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix}$, которая является порождающей матрицей для обобщенных чисел Фибоначчи порядка *m*. Доказано следующее е свойство Q_m-матрицы:

$$G_m^n = \begin{pmatrix} F_m(n+1) & F_m(n) \\ F_m(n) & F_m(n-1) \end{pmatrix}$$

(4) В работе [26] автором введен новый класс квадратных матриц, основанных на использовании симметричных гиперболических функций Фибоначчи (Стахов, Розин, 2005). Исследовании Q_m-матриц привело к открытию нового класса квадратных матриц, основанных на использовании гиперболических функций Фибоначчи порядка *m*:

$$G_m^{2x} = \begin{pmatrix} cF_m(2x+1) & sF_m(2x) \\ sF_m(2x) & cF_m(2x-1) \end{pmatrix} \qquad G_m^{2x+1} = \begin{pmatrix} sF_m(2x+2) & cF_m(2x+1) \\ cF_m(2x+1) & sF_m(2x) \end{pmatrix}$$

(5) В работе [26] автором введен новый криптографический метод, основанный на использовании «золотых» матриц и названный «золотой» криптографией. В настоящей работе предложен усовершенствованный метод «золотой» криптографии, основанный на использовании гиперболических функций Фибоначчи порядка *m*. Новым свойством усовершенствованного метода является наличие двух непрерывных переменных *x* и *m*, которые могут быть использованы в качестве «криптографических ключей», что расширяет возможности криптографической зашиты.

Таким образом, формулы Газале и вытекающие из них новые математические результаты в области гиперболических функций Фибоначчи и Люка и «золотых» матриц, полученные в настоящей работе, открывают интересные перспективы для создания новых гиперболических моделей Природы (теоретическая физика) и новых методов кодирования и криптографии (компьютерные науки).

Abstract

We consider the Gazale formulas, which are a wide generalization of the Binet and Pell formulas, and a new class of the "golden" hyperbolic functions, which a wide generalization of the symmetric hyperbolic Fibonacci and Lucas functions (Stakhov and Rozin, 2005). Also we consider a new class of the "golden" matrices being a wide generalization of the "golden" matrices (Stakhov, 2006). The improved cryptographic method, which is a generalization of Stakhov's "golden" cryptographic method, follows from the new "golden" matrices.

1. Introduction

The present paper is devoted to the generalization and development of a number of fundamental results of the contemporary Fibonacci numbers theory [1-28]. And we will begin from the review of these fundamental notions and concepts.

Fibonacci and Lucas numbers

Consider the following recursive relation

$$F(n) = F(n-1) + F(n-2) \tag{1}$$

where $n = 0, \pm1, \pm2, \pm3, \ldots$. For the seeds

$$F(0) = 0, F(1) = 1 \qquad (2)$$

the recursive relation (1) sets the classical Fibonacci sequence expanded to the side of the negative values of n:

$$\ldots -21, 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots \qquad (3)$$

This sequence is symmetric relative to the number 0, if we take into consideration that every even term of the sequence (3) from the left of the number 0 is negative.

Also we can consider the recursive relation

$$L(n) = L(n-1) + L(n-2) \qquad (4)$$

For the seeds

$$L(0) = 2, L(1) = 1, \qquad (5)$$

the recursive relation (4) sets the classical Lucas sequence expanded to the side of the negative values of n:

$$\ldots 47, -29, 18, -11, 7, -4, 3, -1, 2, 1, 3, 4, 7, 11, 18, 29, 47, \ldots \qquad (6)$$

This sequence is symmetric relative to the number 2, if we take into consideration that every odd term of the sequence (6) from the left of the number 2 is negative.

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Binet formulas

As is known, the recursive relation (1) can be represented in the form:

$$\frac{F(n)}{F(n-1)} = 1 + \frac{1}{\frac{F(n-1)}{F(n-2)}} \qquad (7)$$

For the case $n \to \infty$ the expression (7) is reduced to the following quadratic equation:

$$x^2 - x - 1 = 0 \qquad (8)$$

The equation (8) has two roots:

$$x_1 = \Phi_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad x_2 = -\frac{1}{\Phi_1} = \frac{1-\sqrt{5}}{2}, \qquad (9)$$

They are connected by the following correlation:

$$x_1 + x_2 = 1 \qquad (10)$$

The following equality follows from (10):

$$\Phi_1 - \frac{1}{\Phi_1} = 1 \qquad (11)$$

The roots (9) are the "launching pad" for the derivation of Binet formulas for Fibonacci and Lucas numbers:

$$F_1(n) = \frac{\Phi_1^n - \left(-\frac{1}{\Phi_1}\right)^n}{\sqrt{5}} \qquad (12)$$

$$L_1(n) = \Phi_1^n + \left(-\frac{1}{\Phi_1}\right)^n \qquad (13)$$

The formulas (12), (13) were derived by Binet in 1843, although the result was known to Euler, Daniel Bernoulli, and de Moivre more than a century earlier. In particular, de Moivre derived these formulas in 1718.

Q-matrix

Verner Hoggatt developed in [3] a theory of Fibonacci *Q*-matrix:

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad (14)$$

which is connected with the Fibonacci numbers (3) by the following formula:

$$Q^n = \begin{pmatrix} F(n+1) & F(n) \\ F(n) & F(n-1) \end{pmatrix} \qquad (15)$$

It is proved in [3] that the determinant of the matrix (15) coincides with the famous Cassini formula

$$Det\, Q^n = F(n+1)F(n-1) - F^2(n) = (-1)^n$$

that was named in honor of the well-known 17-th century astronomer Giovanni Cassini (1625-1712) who first derived this formula.

Recently in the works of Alexey Stakhov and Boris Rozin the classical results in the Fibonacci numbers theory were generalized [11-28]. Consider some of these generalizations.

Hyperbolic Fibonacci and Lucas functions

Stakhov and Rozin introduced in [14] the so-called *symmetric hyperbolic Fibonacci and Lucas functions*

Symmetric hyperbolic Fibonacci functions

$$sFs(x) = \frac{\Phi_1^x - \Phi_1^{-x}}{\sqrt{5}} \qquad cFs(x) = \frac{\Phi_1^x + \Phi_1^{-x}}{\sqrt{5}} \qquad (16)$$

Symmetric hyperbolic Lucas functions

$$sLs(x) = \Phi_1^x - \Phi_1^{-x} \qquad cLs(x) = \Phi_1^x + \Phi_1^{-x} \qquad (17)$$

The Fibonacci and Lucas numbers (3), (6) are determined identically through the symmetric hyperbolic Fibonacci and Lucas functions (16), (17) as follows:

$$F(n) = \begin{cases} sFs(n), & for \quad n = 2k \\ cFs(n), & for \quad n = 2k+1 \end{cases}; \qquad L(n) = \begin{cases} cLs(n), & for \quad n = 2k \\ sLs(n), & for \quad n = 2k+1 \end{cases}. \qquad (18)$$

The "golden" Q-matrices and the "golden" cryptography

Stakhov developed in [26] a new class of the square matrices called *"golden" matrices*:

$$Q^{2x} = \begin{pmatrix} cFs(2x+1) & sFs(2x) \\ sFs(2x) & cFs(2x-1) \end{pmatrix} \quad Q^{2x+1} = \begin{pmatrix} sFs(2x+2) & cFs(2x+1) \\ cFs(2x+1) & sFs(2x) \end{pmatrix} \qquad (19)$$

These matrices were used by Stakhov for the development of a new kind of cryptography called the *"golden" cryptography* [26].

The generalized Fibonacci numbers of the order m

In the last years many researchers (Vera W. de Spinadel [8], Jay Kappraff [9], Midhat J. Gazale [10] and others) independently one from another introduced the generalized Fibonacci numbers based on the following recursive relation:

$$F_m(n+2) = mF_m(n+1) + F_m(n) \qquad (20)$$

$$F_m(0) = 0, F_m(1) = 1 \tag{21}$$

where m is a positive real number.

The Egyptian mathematician Midhat J. Gazale in his remarkable book [10] called new recursive sequences based on (20), (21) the *generalized Fibonacci sequence of the order m*.

As example we can see in Table 1 the different Fibonacci sequences $F_m(n)$ of the orders m=1, 2, $\frac{1}{\sqrt{2}}$ for the values n from -4 to 5.

Table 1. The Fibonacci sequences $F_m(n)$

m/n	-4	-3	-2	-1	0	1	2	3	4	5
1	-3	2	-1	1	0	1	1	2	3	5
2	-12	5	-2	1	0	1	2	5	12	29
$1/\sqrt{2}$	$-5/(2\sqrt{2})$	3/2	$-1/(\sqrt{2})$	-1	0	1	$1/\sqrt{2}$	3/2	$5/(2\sqrt{2})$	11/4

We can see from Table 1 that for the case m=1 the Fibonacci sequence $F_1(n)$ coincides with the classical Fibonacci numbers (3) and for the case m=2 the Fibonacci sequence $F_2(n)$ coincides with the Pell numbers.

The main purpose of the present paper is to use the recursive relation (20), (21) for the generalization and development of all the above mathematical results (1)-(21). In particular, basing on (20), (21), we have derived the so-called *Gazale formulas,* then, basing on Gazale formulas, we have introduced a *new class of the hyperbolic Fibonacci and Lucas functions*, which are a generalization of (16), (17). Also basing on new hyperbolic Fibonacci and Lucas functions, we have developed the *improved method of the "golden" cryptography*.

2. Gazale formulas

Represent the recursive relation (1) in the form:

$$\frac{F_1(n+2)}{F(n+1)} = m + \frac{1}{\frac{F(n+1)}{F(n)}} \tag{22}$$

For the case $n \to \infty$ the expression (22) is reduced to the following quadratic equation:

$$x^2 - mx - 1 = 0 \tag{23}$$

The equation (23) has two roots, a positive root

$$x_1 = \frac{\sqrt{4+m^2}+m}{2} \tag{24}$$

and a negative root

$$x_2 = \frac{-\sqrt{4+m^2}+m}{2} \tag{25}$$

If we sum term-wise the roots (24) and (25) we will get:

$$x_1 + x_2 = m \tag{26}$$

If we substitute the roots (24), (25) into Eq. (23) instead x, we will get the following identities:

$$x_1^2 = mx_1 + 1 \tag{27}$$

$$x_2^2 = mx_2 + 1 \tag{28}$$

If we multiply or divide repeatedly all terms of the identities (27) and (28) by x_1 and x_2, respectively, we will get the following identities:

$$x_1^n = mx_1^{n-1} + x_1^{n-2}, \tag{29}$$

$$x_2^n = mx_2^{n-1} + x_2^{n-2} \tag{30}$$

where n=0, ±1, ±2, ±3, … .

Gazale denoted the positive root x_1 by Φ_m and named it a *"start point number"* and the number m a *"gnomonic number"* of the number Φ_m. A sense of such definition becomes clear below.

The "start point number" Φ_m has the following analytical expression:

$$\Phi_m = \frac{\sqrt{4+m^2}+m}{2} \tag{31}$$

Let us express now the root x_2 through the "start point number" Φ_m. After simple transformation of (25) we can write the root x_2 as follows:

$$x_2 = \frac{-\sqrt{4+m^2}+m}{2} = \frac{-4}{2(\sqrt{4+m^2}+m)} = -\frac{1}{\Phi_m} \tag{32}$$

Substituting Φ_m instead x_1 and $-\frac{1}{\Phi_m}$ instead x_2 in (26) we will get the following identity:

$$m = \Phi_m - \frac{1}{\Phi_m}, \tag{33}$$

where Φ_m is given by (31) and $\frac{1}{\Phi_m}$ is given by the formula:

$$\frac{1}{\Phi_m} = \frac{\sqrt{4+m^2}-m}{2} \tag{34}$$

Notice that for the case m=1 the formula (31) is reduced to the classical golden ratio $\Phi_1 = \frac{1+\sqrt{5}}{2}$. Basing on this fact we will name Gazale's "start point number" Φ_m a *generalized golden ratio of the order m.*

Write the obvious property of the generalized golden ratio of the order m:

$$\Phi_m + \frac{1}{\Phi_m} = \sqrt{4+m^2} \tag{35}$$

Using the identity (29) we can write the following identity for the number Φ_m:

$$\Phi_m^n = m\Phi_m^{n-1} + \Phi_m^{n-2}, \tag{36}$$

where n=0, ±1, ±2, ±3, … .

Two surprising representations of the generalized golden ratio Φ_m:

For the case n=2 the identity (36) can be represented in the form:

$$\Phi_m^2 = 1 + m\Phi_m \tag{37}$$

Basing on (37) we can write the following representation of the generalized golden ratio Φ_m:

$$\Phi_m = \sqrt{1+m\Phi_m} \tag{38}$$

Substituting instead Φ_m in the right-hand part of (38) the same expression (38) we can write:

$$\Phi_m = \sqrt{1+m\sqrt{1+\Phi_m}} \tag{39}$$

Continuing this process ad infinitum, that is, substituting repeatedly instead Φ_m in the right-hand part of (39) the expression (38), we can get the following surprising representation of the generalized golden ratio Φ_m:

$$\Phi_m = \sqrt{1 + m\sqrt{1 + m\sqrt{1 + m\sqrt{\ldots}}}} \quad (40)$$

Represent now the identity (37) in the form:

$$\Phi_m = m + \frac{1}{\Phi_m} \quad (41)$$

Substituting instead Φ_m in the right-hand part of (41) the same expression (41) we can write:

$$\Phi_m = m + \frac{1}{m + \frac{1}{\Phi_m}} \quad (42)$$

Continuing this process ad infinitum, that is, substituting repeatedly instead Φ_m in the right-hand part of (42) the expression (41), we can get the following surprising representation of the generalized golden ratio Φ_m:

$$\Phi_m = m + \frac{1}{m + \frac{1}{m + \frac{1}{m + \ldots}}} \quad (43)$$

A derivation of Gazale formula

In the formulas (20), (21) the numbers $F_m(n)$ are defined by recursion. We can express the numbers $F_m(n)$ in explicit form by using the generalized golden ratio Φ_m.

We will look for the analytical expression of the generalized Fibonacci number $F_m(n)$ through the roots x_1 and x_2 in the form:

$$F_m(n) = k_1 x_1^n + k_2 x_2^n \quad (44)$$

where k_1 and k_2 are constant coefficients, which are solutions of the following system of algebraic equation:

$$\begin{cases} F_m(0) = k_1 x_1^0 + k_2 x_2^0 = k_1 + k_2 \\ F_m(1) = k_1 x_1^1 + k_2 x_2^1 = k_1 \Phi_m - k_2 \frac{1}{\Phi_m} \end{cases} \quad (45)$$

Taking into consideration that $F_m(0)=0$ and $F_m(1)=1$ we can rewrite the system (45) as follows:

$$k_1 = - k_2 \quad (46)$$

and

$$k_1 \Phi_m + k_1 \frac{1}{\Phi_m} = k_1 \left(\Phi_m + \frac{1}{\Phi_m} \right) = 1 \quad (47)$$

Taking into consideration (46) and (47), we can find the following expressions for the coefficients k_1 and k_2:

$$k_1 = \frac{1}{\sqrt{4 + m^2}}\,;\; k_2 = -\frac{1}{\sqrt{4 + m^2}} \quad (48)$$

Taking into consideration the expressions (48) we can write the expression (44) as follows:

$$F_m(n) = \frac{1}{\sqrt{4 + m^2}} x_1^n - \frac{1}{\sqrt{4 + m^2}} x_2^n = \frac{1}{\sqrt{4 + m^2}} \left(x_1^n - x_2^n \right) \quad (49)$$

Taking into consideration that $x_1 = \Phi_m$ and $x_2 = -\frac{1}{\Phi_m}$, we can rewrite the formula (49) as follows:

$$F_m(n) = \frac{\Phi_m^n - (-1/\Phi_m)^n}{\sqrt{4+m^2}} \qquad (50)$$

or

$$F_m(n) = \frac{1}{\sqrt{4+m^2}}\left[\left(\frac{m+\sqrt{4+m^2}}{2}\right)^n - \left(\frac{m-\sqrt{4+m^2}}{2}\right)^n\right] \qquad (51)$$

For the partial case m=1 the formula (51) is reduced to the Binet formula (12).

For the case m=2 the formula (50) takes the following form:

$$F_2(n) = \frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n\right] \qquad (52)$$

Notice that for the first time this formula was derived by the English mathematician John Pell (1610-1685).

For the case m=3 and $m=\sqrt{2}$ the formula (51) takes the following forms, respectively:

$$F_3(n) = \frac{1}{\sqrt{13}}\left[\left(\frac{3+\sqrt{13}}{2}\right)^n - \left(\frac{3-\sqrt{13}}{2}\right)^n\right] \qquad (53)$$

$$F_{\sqrt{2}}(n) = \frac{1}{\sqrt{6}}\left[\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right)^n - \left(\frac{\sqrt{2}-\sqrt{6}}{2}\right)^n\right] \qquad (54)$$

Thus, the Egyptian mathematician Midhat J. Gazale has derived recently the unique mathematical formula (51), which includes as partial cases Binet formula for Fibonacci numbers (12) for the case m=1 and Pell formula (52) for the case m=2. However, this formula generates an infinite number of the generalized Fibonacci numbers of the order m because m is positive real number. Due uniqueness of the formula (51) we will name this formula *Gazale formula for the generalized Fibonacci numbers of the order m* or simply *Gazale formula.*

3. The generalized Fibonacci and Lucas numbers of the order *m*

The generalized Fibonacci numbers of the order m

Let us prove that Gazale formulas (51)-(54) really expresses all generalized Fibonacci numbers of the order m given by the recursive formula (20) at the seeds (21). In fact, for the case n=0 it follows directly from the formula (51) that $F_m(0)=0$. For the case n=1 we can write the formula (51) as follows:

$$F_m(1) = \frac{1}{\sqrt{4+m^2}}\left(\frac{m+\sqrt{4+m^2}}{2} - \frac{m-\sqrt{4+m^2}}{2}\right) = 1.$$

This means that the formula (51) corresponds to the seeds (21).

Suppose that the formula (49) is valid for a given n (the inductive hypothesis) and prove that this formula is valid for the case n+1, that is,

$$F_m(n+1) = \frac{1}{\sqrt{4+m^2}}\left(x_1^{n+1} - x_2^{n+1}\right) \qquad (55)$$

Using the identities (24) and (25) we can represent the formula (55) as follows:

$$F_m(n+1) = \frac{m}{\sqrt{4+m^2}}\left(x_1^n - x_2^n\right) + \frac{1}{\sqrt{4+m^2}}\left(x_1^{n-1} - x_2^{n-1}\right) = mF_m(n) + F_m(n-1) \quad (56)$$

Thus, the formula (56), in fact, sets the generalized Fibonacci numbers of the order m given by the recursive relation (20), (21).

Notice that the formula (51) sets all generalized Fibonacci numbers $F_m(n)$ in the range n=0, ±1, ±2, ±3, … . Let us find some surprising properties of the generalized Fibonacci numbers of the order m. First of all we can compare the generalized Fibonacci numbers $F_m(n)$ and $F_m(-n)$. We can write the formula (51) as follows:

$$F_m(n) = \frac{\Phi_m^n - (-1)^n \Phi_m^{-n}}{\sqrt{4+m^2}} \quad (57)$$

Represent now the formula (57) for the negative values of n, that is,

$$F_m(-n) = \frac{\Phi_m^{-n} - (-1)^{-n} \Phi_m^{n}}{\sqrt{4+m^2}} \quad (58)$$

Comparing the expression (57) and (58) for the even (n=2k) and odd (n=2k+1) values of n, we can conclude that

$$F_m(2k) = -F_m(-2k) \text{ and } F_m(2k+1) = F_m(-2k-1). \quad (59)$$

This means that the sequences of the generalized Fibonacci numbers of the order m in the range n=0, ±1, ±2, ±3, … is a symmetric sequence relative to the generalized Fibonacci number $F_m(0) = 0$ excepting that the generalized Fibonacci numbers $F_m(2k)$ and $F_m(-2k)$ are opposite by sign.

In Table 2 we can see the generalized Fibonacci numbers with the orders m=1, 2, 3, 4

Table 2. The generalized Fibonacci sequences with the orders m=1, 2, 3, 4

m	Φ_m	-5	-4	-3	-2	-1	0	1	2	3	4	5
1	$\frac{1+\sqrt{5}}{2}$	5	-3	2	-1	1	0	1	1	2	3	5
2	$1+\sqrt{2}$	29	-12	5	-2	1	0	1	2	5	12	29
3	$\frac{3+\sqrt{13}}{2}$	109	-33	10	-3	1	0	1	3	10	33	109
4	$2+\sqrt{5}$	305	-72	17	-4	1	0	1	4	17	72	305

Notice that for the case m=2 the Gazele formula (57) generates a numerical sequence known as *Pell numbers.*

Let us find the fundamental formula, which connects the three adjacent generalized Fibonacci numbers with the order m. For the case m=1 this formula is known as Cassini formula. We can represent this formula for the classical Fibonacci numbers $F_1(n)$ as follows:

$$F_1^2(n) - F_1(n-1)F_1(n+1) = (-1)^{n+1} \quad (60)$$

It is easy to prove the following general identity for the generalized Fibonacci numbers of the order m:

$$F_m^2(n) - F_m(n-1)F_m(n+1) = (-1)^{n+1} \quad (61)$$

For example, for the case m=2 the Fibonacci numbers $F_2(-5)$=29, $F_2(-4)$=-12 and $F_2(-3)$=5 are connected by the following correlation: $(-12)^2$ - 29×5= -1, and for the case m=3 the Fibonacci numbers $F_3(4)$=33, $F_3(3)$=10 and $F_3(3)$=3 are connected by the following correlation: $(10)^2$ - 33×3 = 1.

The generalized Lucas numbers of the order m

Consider once again the formula (44) given the generalized Fibonacci numbers of the order m.

By analogy to the classical Lucas numbers we can consider the formula

$$L_m(n) = x_1^n + x_2^n \tag{62}$$

It is clear that for the case m=1 this formula sets the classical Lucas numbers (6). We will assume that the formula (62) sets *the generalized Lucas numbers of the order m.* For a given m we can find some peculiarities of the generalized Lucas numbers of the order m. First of all we can calculate the seeds of the generalized Lucas numbers of the order m. In fact, for the case n=0 and n=1 we have respectively:

$$L_m(0) = x_1^0 + x_2^0 = 1 + 1 = 2\,; \tag{63}$$

$$L_m(1) = x_1^1 + x_2^1 = m \tag{64}$$

Using the identities (29) and (30) we can represent the formula (62) as follows:

$$L_m(n) = x_1^n + x_2^n = mx_1^{n-1} + x_2^{n-2} + mx_1^{n-2} + x_2^{n-2} = m\left(x_1^{n-1} + x_2^{n-1}\right) + \left(x_1^{n-2} + x_2^{n-2}\right) \tag{65}$$

Taking into consideration the definition (62) we can write (65) in the form of the following recursive relation:

$$L_m(n) = mL_m(n-1) + L_m(n-2) \tag{66}$$

It is clear that the recursive relation (66) at the seeds (63), (64) sets the generalized Lucas numbers of the order m.

If we substitute in the formula (62) instead x_1 and x_2 their expressions through the generalized golden ratio of the order m $x_1 = \Phi_m$ and $x_2 = -\dfrac{1}{\Phi_m}$ we can represent the formula (62) as follows:

$$L_m(n) = \left[\Phi_m^n + \left(\frac{-1}{\Phi_m}\right)^n\right] \tag{67}$$

Although this formula is absent in the book [9] we will name this important formula *Gazale formula for the generalized Lucas numbers of the order m.*

We can rewrite the formula (67) as follows:

$$L_m(n) = \Phi_m^n + (-1)^n \Phi_m^{-n} \tag{68}$$

Represent now the formula (68) for the negative values of n, that is,

$$L_m(-n) = \Phi_m^{-n} + (-1)^{-n} \Phi_m^n \tag{69}$$

Comparing the expression (68) and (69) for the even (n=2k) and odd (n=2k+1) values of n, we can conclude that

$$L_m(2k) = L_m(-2k) \text{ and } L_m(2k+1) = - F_m(-2k-1). \tag{70}$$

This means that the sequences of the generalized Lucas numbers of the order m in the range n=0, ±1, ±2, ±3, … is a symmetrical sequence relative to the generalized Lucas number $L_m(0)$ = 2 excepting that the generalized Lucas numbers $L_m(2k+1)$ and $L_m(-2k-1)$ are opposite by sign.

In Table 3 we can see the generalized Lucas numbers with the orders m=1, 2, 3, 4.

Table 3. The generalized Lucas sequences with the orders m=1, 2, 3, 4

m	Φ_m	-5	-4	-3	-2	-1	0	1	2	3	4	5
1	$\frac{1+\sqrt{5}}{2}$	-11	7	-4	3	-1	2	1	3	4	7	11
2	$1+\sqrt{2}$	-82	34	-14	6	-2	2	2	6	14	34	82
3	$\frac{3+\sqrt{13}}{2}$	-393	119	-36	11	-3	2	3	11	36	119	393
4	$2+\sqrt{5}$	-1364	322	-76	18	-4	2	4	18	76	322	1364

Notice that for the case m=2 the Gazele formula (69) generates a numerical sequence known as *Pell-Lucas numbers.*

4. A new class of the "golden" hyperbolic functions

A definition of the hyperbolic Fibonacci and Lucas functions of the order m

Stakhov and Rozin introduced in [14] a new class of the hyperbolic functions, the symmetric hyperbolic Fibonacci and Lucas functions, basing on an analogy between Binet formulas (12), (13) and the classical hyperbolic functions. We can use this approach to introduce the hyperbolic Fibonacci and Lucas functions of the order m basing on an analogy between Gazele formulas given by (57) and (68) and the classical hyperbolic functions.

Hyperbolic Fibonacci sine of the order m

$$sF_m(x)=\frac{\Phi_m^x-\Phi_m^{-x}}{\sqrt{4+m^2}}=\frac{1}{\sqrt{4+m^2}}\left[\left(\frac{m+\sqrt{4+m^2}}{2}\right)^x-\left(\frac{m+\sqrt{4+m^2}}{2}\right)^{-x}\right] \tag{71}$$

Hyperbolic Fibonacci cosine of the order m

$$cF_m(x)=\frac{\Phi_m^x+\Phi_m^{-x}}{\sqrt{4+m^2}}=\frac{1}{\sqrt{4+m^2}}\left[\left(\frac{m+\sqrt{4+m^2}}{2}\right)^x+\left(\frac{m+\sqrt{4+m^2}}{2}\right)^{-x}\right] \tag{72}$$

Hyperbolic Lucas sine of the order m

$$sL_m(x)=\Phi_m^x-\Phi_m^{-x}=\left(\frac{m+\sqrt{4+m^2}}{2}\right)^x-\left(\frac{m+\sqrt{4+\sqrt{4+m^2}}}{2}\right)^{-x} \tag{73}$$

Hyperbolic Lucas cosine of the order m

$$cL_m(x)=\Phi_m^x+\Phi_m^{-x}=\left(\frac{m+\sqrt{4+m^2}}{2}\right)^x+\left(\frac{m+\sqrt{4+\sqrt{4+m^2}}}{2}\right)^{-x} \tag{74}$$

The generalized Fibonacci and Lucas numbers of the order m are determined identically through the hyperbolic Fibonacci and Lucas functions of the order m as follows:

$$F_m(n)\;=\;\begin{cases}sF_m(n), & for\quad n=2k\\ cF_m(n), & for\quad n=2k+1\end{cases};\qquad L_m(n)\;=\;\begin{cases}cL_m(n), & for\quad n=2k\\ sL_m(n), & for\quad n=2k+1\end{cases}. \tag{75}$$

The graphs of the hyperbolic Fibonacci and Lucas functions of the order m are similar to the graphs of the classical hyperbolic functions. Here is necessity to notice that in the point x=0, the hyperbolic Fibonacci cosine $cF_m(x)$ (72) takes the value $cF_m(0)=\frac{2}{\sqrt{4+m^2}}$, and the hyperbolic Lucas cosine $cL_m(x)$ (74) takes the value $cL_m(0)$ = 2. It is also important to emphasize that the generalized Fibonacci numbers $F_m(n)$ with the even values of n = 0, ±2, ±4, ±6, … are "inscribed" into the graph of the hyperbolic Fibonacci sine $sF_m(x)$ in the discrete points x = 0, ±2, ±4, ±6, … and the generalized Fibonacci numbers $F_m(n)$ with the odd values of n = ±1, ±3, ±5, … are "inscribed" into the hyperbolic Fibonacci cosine $cF_m(x)$ in the discrete points x = ±1, ±3, ±5 …. In the other hand, the generalized Lucas numbers $L_m(n)$ with the even values of n are "inscribed" into the graph of the hyperbolic Lucas cosine $cL_m(x)$ in the discrete points x = 0, ±2, ±4, ±6 …, and the generalized Lucas numbers $L_m(n)$ with the odd values of n are "inscribed" into the graph of the hyperbolic Lucas sine $sL_m(x)$ in the discrete points $x = \pm1, \pm3, \pm5$ ….

Also we can introduce the notions of the hyperbolic Fibonacci and Lucas tangents and cotangents of the order m.

Hyperbolic Fibonacci tangent of the order m

$$tF_m(x) = \frac{sF_m(x)}{cF_m(x)} = \frac{\Phi_m^x - \Phi_m^{-x}}{\Phi_m^x + \Phi_m^{-x}} \quad (76)$$

Hyperbolic Fibonacci cotangent of the order m

$$ctF_m(x) = \frac{cF_m(x)}{sF_m(x)} = \frac{\Phi_m^x + \Phi_m^{-x}}{\Phi_m^x - \Phi_m^{-x}} \quad (77)$$

Hyperbolic Lucas tangent of the order m

$$tL_m(x) = \frac{sL_m(x)}{cL_m(x)} = \frac{\Phi_m^x - \Phi_m^{-x}}{\Phi_m^x + \Phi_m^{-x}} \quad (78)$$

Hyperbolic Lucas cotangent of the order m

$$ctL_m(x) = \frac{cL_m(x)}{sL_m(x)} = \frac{\Phi_m^x + \Phi_m^{-x}}{\Phi_m^x - \Phi_m^{-x}} \quad (79)$$

By analogy we can introduce other hyperbolic Fibonacci and Lucas functions of the order *m,* in particular, secant and cosecant and so on.

Properties of the hyperbolic Fibonacci and Lucas functions of the order m

It easy to prove that the function (71) is an odd function because

$$sF_m(-x) = \frac{\Phi_m^{-x} - \Phi_m^x}{\sqrt{4+m^2}} = -\frac{\Phi_m^x - \Phi_m^{-x}}{\sqrt{4+m^2}} = -sF_m(x) \quad (80)$$

On the other hand,

$$cF_m(-x) = \frac{\Phi_m^{-x} + \Phi_m^x}{\sqrt{4+m^2}} = -\frac{\Phi_m^x + \Phi_m^{-x}}{\sqrt{4+m^2}} = cF_m(x) \quad (81)$$

that is, the hyperbolic Fibonacci cosine (72) is an even function.

By analogy we can prove that the hyperbolic Lucas sine of the order m (73) is an odd function and the hyperbolic Lucas cosine of the order m (74) is an even function.

Making the pair-wise comparison of the functions (76) and (78), (77) and (79) we can conclude that the functions of the hyperbolic Fibonacci and Lucas tangents and cotangents of the order m are coincident, that is, we have:

$$tF_m(x) = tL_m(x) \text{ and } ctF_m(x) = ctL_m(x). \quad (82)$$

It is easy to prove that the functions (76), (77) are the odd functions because

$$tF_m(-x) = \frac{\Phi_m^{-x} - \Phi_m^x}{\Phi_m^{-x} + \Phi_m^x} = -tF_m(x)$$

$$ctF_m(-x)=\frac{\Phi_m^{-x}+\Phi_m^{x}}{\Phi_m^{-x}-\Phi_m^{x}}=-ctF_m(x)$$

Taking into consideration (82) we can write:

$$tL_m(-x)=-tL_m(x)$$

$$ctL_m(-x)=-ctL_m(x)$$

that is, the functions (78), (79) are the odd functions too.

Thus, we have introduced above very interesting class of the hyperbolic functions, which are a wide generalization of the symmetric hyperbolic Fibonacci and Lucas functions (16), (17), which are a partial case of the above hyperbolic functions given by (71)-(74) for the case m=1. Let us consider the analytical expressions of the hyperbolic Fibonacci and Lucas functions (71)-(74) for the different values of the order m.

Hyperbolic Fibonacci and Lucas functions of the order m=1

$$sF_1(x)=\frac{\Phi_1^{x}-\Phi_1^{-x}}{\sqrt{5}}=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{x}-\left(\frac{1+\sqrt{5}}{2}\right)^{-x}\right] \quad (83)$$

$$csF_1(x)=\frac{\Phi_1^{x}+\Phi_1^{-x}}{\sqrt{5}}=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{x}+\left(\frac{1+\sqrt{5}}{2}\right)^{-x}\right] \quad (84)$$

$$sL_1(x)=\Phi_1^{x}-\Phi_1^{-x}=\left(\frac{1+\sqrt{5}}{2}\right)^{x}-\left(\frac{1+\sqrt{5}}{2}\right)^{-x} \quad (85)$$

$$csL_1(x)=\Phi_1^{x}-\Phi_1^{-x}=\left(\frac{1+\sqrt{5}}{2}\right)^{x}+\left(\frac{1+\sqrt{5}}{2}\right)^{-x} \quad (86)$$

Hyperbolic Fibonacci and Lucas functions of the order m=2

$$sF_2(x)=\frac{\Phi_2^{x}-\Phi_2^{-x}}{\sqrt{8}}=\frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^{x}-\left(1+\sqrt{2}\right)^{-x}\right] \quad (87)$$

$$cF_2(x)=\frac{\Phi_2^{x}+\Phi_2^{-x}}{\sqrt{8}}=\frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^{x}+\left(1+\sqrt{2}\right)^{-x}\right] \quad (88)$$

$$sL_2(x)=\Phi_2^{x}-\Phi_2^{-x}=\left(1+\sqrt{2}\right)^{x}-\left(1+\sqrt{2}\right)^{-x} \quad (89)$$

$$cL_2(x)=\Phi_2^{x}+\Phi_2^{-x}=\left(1+\sqrt{2}\right)^{x}+\left(1+\sqrt{2}\right)^{-x} \quad (90)$$

Hyperbolic Fibonacci and Lucas functions of the order m=3

$$sF_3(x)=\frac{\Phi_3^{x}-\Phi_3^{-x}}{\sqrt{13}}=\frac{1}{\sqrt{13}}\left[\left(\frac{3+\sqrt{13}}{2}\right)^{x}-\left(\frac{3+\sqrt{13}}{2}\right)^{-x}\right] \quad (91)$$

$$cF_3(x)=\frac{\Phi_3^{x}+\Phi_3^{-x}}{\sqrt{13}}=\frac{1}{\sqrt{13}}\left[\left(\frac{3+\sqrt{13}}{2}\right)^{x}+\left(\frac{3+\sqrt{13}}{2}\right)^{-x}\right] \quad (92)$$

$$sL_3(x) = \Phi_3^x - \Phi_3^{-x} = \left(\frac{3+\sqrt{13}}{2}\right)^x - \left(\frac{3+\sqrt{13}}{2}\right)^{-x} \quad (93)$$

$$sL_3(x) = \Phi_3^x + \Phi_3^{-x} = \left(\frac{3+\sqrt{13}}{2}\right)^x + \left(\frac{3+\sqrt{13}}{2}\right)^{-x} \quad (94)$$

Notice that a list of these functions can be continued ad infinitum.

It is easy to see that the functions (71)-(74) are connected by very simple correlation:

$$sF_m(x) = \frac{sL_m(x)}{\sqrt{4+m^2}} \quad ; \quad cF_m(x) = \frac{cL_m(x)}{\sqrt{4+m^2}} \quad . \quad (95)$$

This means that the hyperbolic Lucas functions of the order *m* (73), (74) coincide with the hyperbolic Fibonacci functions of the order (71), (72) to within of the constant coefficient $\frac{1}{\sqrt{1+m^2}}$. Taking into consideration this fact we will consider in further only the hyperbolic Fibonacci functions of the order *m*.

We can prove a number of the important theorem for the hyperbolic Fibonacci and Lucas functions of the order *m*.

The hyperbolic Fibonacci and Lucas functions of the order *m* possess the recursive properties similar to the generalized Fibonacci and Lucas numbers of the order *m* given by the recursive relations (20) and (66). On the other hand, they possess all hyperbolic properties similar to the properties of the classical hyperbolic functions. Prove the recursive and hyperbolic properties of the hyperbolic Fibonacci and Lucas functions of the order *m*.

Theorem 1. The following correlations that are analogous to the recurrent relation for the generalized Fibonacci numbers $F_m(n+2) = mF_m(n+1) + F_m(n)$ are valid for the hyperbolic Fibonacci functions of the order *m*:

$$sF_m(x+2) = mcF_m(x+1) + sF_m(x) \quad (96)$$

$$cF_m(x+2) = msF_m(x+1) + cF_m(x)\,. \quad (97)$$

Proof:

$$mcF_m(x+1) + sF_m(x) = m\frac{\Phi_m^{x+1}+\Phi_m^{-x-1}}{\sqrt{4+m^2}} + \frac{\Phi_m^x - \Phi_m^{-x}}{\sqrt{4+m^2}} = \frac{\Phi_m^x(m\Phi_m+1) - \Phi_m^{-x}(1-m\Phi_m^{-1})}{\sqrt{4+m^2}} \quad (98)$$

Because $m\Phi_m + 1 = \Phi_m^2$ and $1-\Phi_m^{-1} = \Phi_m^{-2}$ we can represent (98) as follows:

$$mcF_m(x+1) + sF_m(x) = \frac{\Phi_m^{x+2} - \Phi_m^{-x-2}}{\sqrt{4+m^2}} = sF_m(x+2)$$

that proves the identity (96).

By analogy we can prove the identity (97).

Theorem 2 (a generalization of Cassini formula). The following correlations that are similar to the Cassini formula $F_m^2(n) - F_m(n-1)F_m(n+1) = (-1)^{n+1}$ are valid for the hyperbolic Fibonacci functions of the order *m*:

$$[sFs(x)]^2 - cFs(x+1)\,cFs(x-1) = -1 \tag{99}$$

$$[cFs(x)]^2 - sFs(x+1)\,sFs(x-1) = 1. \tag{100}$$

Proof:

$$[sF_m(x)]^2 - cF_m(x+1)\,cF_m(x-1) = \left(\frac{\Phi_m^x - \Phi_m^{-x}}{\sqrt{4+m^2}}\right)^2 - \frac{\Phi_m^{x+1} + \Phi_m^{-x-1}}{\sqrt{4+m^2}} \times \frac{\Phi_m^{x-1} + \Phi_m^{-x+1}}{\sqrt{4+m^2}} =$$

$$= \frac{\Phi_m^{2x} - 2 + \Phi_m^{-2x} - \left(\Phi_m^{2x} + \Phi_m^{2} + \Phi_m^{-2} + \Phi_m^{-2x}\right)}{4+m^2} = \frac{-2 - \left(\Phi_m^{2} + \Phi_m^{-2}\right)}{4+m^2} \tag{101}$$

Using the formula (68) for the case n=2 we can write:

$$L_m(2) = \Phi_m^2 + \Phi_m^{-2} \tag{102}$$

Using the recursive formula (66) and the seeds (63), (64) we can represent the generalized Lucas number $L_m(2)$ as follows:

$$L_m(2) = mL_m(1) + L_m(0) = m \times m + 2 = m^2 + 2 \tag{103}$$

Taking into consideration (103) we can conclude from (101) that the identity (99) is valid.

By analogy we can prove the identity (100).

Theorem 3. The following identity similar to the identity for the classical hyperbolic functions $[ch(x)]^2 - [sh(x)]^2 = 1$ is valid for the hyperbolic Fibonacci functions of the order m:

$$[cF_m(x)]^2 - [sF_m(x)]^2 = \frac{4}{4+m^2}\ . \tag{104}$$

Proof:

$$[cF_m(x)]^2 - [sF_m(x)]^2 = \left(\frac{\Phi_m^x + \Phi_m^{-x}}{\sqrt{4+m^2}}\right)^2 - \left(\frac{\Phi_m^x - \Phi_m^{-x}}{\sqrt{4+m^2}}\right)^2 =$$

$$= \frac{\Phi_m^{2x} + 2 + \Phi_m^{-2x} - \Phi_m^{2x} + 2 - \Phi_m^{-2x}}{4+m^2} = \frac{4}{4+m^2}$$

$$= \left(\frac{\tau^x + \tau^{-x}}{\sqrt{5}}\right)^2 - \left(\frac{\tau^x - \tau^{-x}}{\sqrt{5}}\right)^2 = \frac{\tau^{2x} + 2 + \tau^{-2x} - \tau^{2x} + 2 - \tau^{-2x}}{5} = \frac{4}{5}$$

By analogy we can prove the following theorem for the hyperbolic Lucas functions of the order m.

Theorem 4. The following identity similar to the identity for the classical hyperbolic functions $[ch(x)]^2 - [sh(x)]^2 = 1$ is valid for the hyperbolic Lucas functions of the order m:

$$[cLs(x)]^2 - [sLs(x)]^2 = 4 \tag{105}$$

Theorem is proved by analogy to Theorem 3.

Theorem 5. The following identity similar to the identity for the classical hyperbolic functions $ch(x+y) = ch(x)ch(y) + sh(x)sh(y)$ is valid for the hyperbolic Fibonacci functions of the order m:

$$\frac{2}{\sqrt{4+m^2}}\, cF_m(x+y) = cF_m(x)cF_m(y) + sF_m(x)sF_m(y)\ . \tag{106}$$

Proof:

$$cF_m(x)cF_s(y) + sF_m(x)sF_m(y) =$$

$$= \frac{\Phi_m^x + \Phi_m^{-x}}{\sqrt{4+m^2}} \times \frac{\Phi_m^y + \Phi_m^{-y}}{\sqrt{4+m^2}} + \frac{\Phi_m^x - \Phi_m^{-x}}{\sqrt{4+m^2}} \times \frac{\Phi_m^y - \Phi_m^{-y}}{\sqrt{4+m^2}} =$$

$$=\frac{\Phi_m^{x+y}+\Phi_m^{x-y}+\Phi_m^{-x+y}+\Phi_m^{-x-y}+\Phi_m^{x+y}-\Phi_m^{x-y}-\Phi_m^{-x+y}+\Phi_m^{-x+y}}{4+m^2}=$$

$$=\frac{2\left(\Phi_m^{x+y}+\Phi_m^{-x-y}\right)}{\sqrt{4+m^2}\times\sqrt{4+m^2}}=\frac{2}{\sqrt{4+m^2}}cF_m(x+y)$$

Theorem 6. The following identity similar to the identity for the classical hyperbolic functions $ch(x-y)=ch(x)ch(y) - sh(x)sh(y)$ is valid for the hyperbolic Fibonacci functions of the order m:

$$\frac{2}{\sqrt{4+m^2}}cF_m(x\text{-}y) = cF_m(x)cF_m(y) - sF_m(x)sF_m(y)\ . \tag{107}$$

Theorem is proved by analogy to Theorem 5.

By analogy we can prove the following theorems for the hyperbolic Fibonacci and Lucas functions of the order m**.**

Theorem 7. The following identities similar to the identity for the classical hyperbolic functions $ch(2x) = [ch(x)]^2 + [sh(x)]^2$ are valid for the hyperbolic Fibonacci and Lucas functions of the order m:

$$\frac{2}{\sqrt{5}}cF_m(2x) = [cF_m(x)]^2 + [sF_m(x)]^2 \tag{108}$$

$$2cL_m(2x) = [cL_m(x)]^2 + [sL_m(x)]^2 \tag{109}$$

Theorem 8. The following identities similar to the identity for the classical hyperbolic functions $sh(2x) = 2sh(x)ch(x)$ are valid for the hyperbolic Fibonacci and Lucas functions of the order m:

$$\frac{1}{\sqrt{4+m^2}}sF_m(2x) = sF_m(x)cF_s(x) \tag{110}$$

$$sL_m(2x) = sL_m(x)cL_s(x) \tag{111}$$

Theorem 9. The following formulas similar to Moivre's formulas for the classical hyperbolic functions $[ch(x)+ sh(x)]^n = ch(nx) + sh(nx)$ are valid for the hyperbolic Fibonacci and Lucas functions of the order m:

$$[cF_m(x) \pm sF_m(x)]^n = \left(\frac{2}{\sqrt{4+m^2}}\right)^{n-1}[cF_m(nx) \pm sF_m(nx)] \tag{112}$$

$$[cL_m(x) \pm sL_m(x)]^n = 2^{n-1}[cF_m(nx) \pm sF_m(nx)] \tag{113}$$

Thus, our investigations led us to the discovery of a unique class of the hyperbolic functions based on the Gazale formulas (57), (68). Remind that during many centuries the science, in particular, mathematics and theoretical physics, used widely the classical hyperbolic functions with the base e. These functions were used by Lobachevsky in his non-Euclidean geometry and by Minkovsky in his geometric interpretation of Einstein's theory of relativity. Stakhov, Tkachenko and Rozin's works [13, 14] violated a monopoly of the classical hyperbolic functions in contemporary mathematics and theoretical physics. Stakhov, Tkachenko and Rozin proved that the geometry of the Living Nature (in particular, botanic phenomenon of phyllotaxis) can be modelled by the hyperbolic Fibonacci and Lucas functions with the base $\Phi_1=\frac{1+\sqrt{5}}{2}$ (the golden ratio). It is clear that the above hyperbolic Fibonacci and Lucas functions of the order m based on the Gazale formulas extend indefinitely a number of new hyperbolic models of Nature. It is difficult to imagine, that the number of new hyperbolic functions given by formulas (71)-(74) is so much, how many exists real numbers! And all of them possess unique recursive and hyperbolic properties similar to the properties of the classical hyperbolic functions and the hyperbolic Fibonacci and Lucas functions introduced in [13, 14]. This fact is of great importance for the development of the contemporary hyperbolic geometry and theoretical physics.

5. Fibonacci G_m-matrices of the order *m*

The "direct" Fibonacci G_m-matrices

The prominent American mathematician Verner Hoggatt, the founder of the Fibonacci Association, developed in his book [3] a theory of the Fibonacci Q-matrix (14), which is a generating matrix for the classical Fibonacci numbers (3). By analogy to (14) we can introduce the G_m-matrix of the order m being a generating matrix for the generalized Fibonacci numbers of the order m given by the recursive relation (20) at the seeds (21).

The G_m-matrix of the order m

$$G_m = \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix} \tag{114}$$

Notice that the determinant of the G_m-matrix (114) is equal:

$$\text{Det } G_m = m\times 0 - 1\times 1 = -1. \tag{115}$$

The following theorem sets a connection of the G_m-matrix (114) with the generalized Fibonacci numbers of the order m given by (20), (21) .

Theorem 9. For a given integer n (n=0, ±1, ±2, ±3, …) the n^{th} power of the G_m-matrix of the order m is given by

$$G_m^n = \begin{pmatrix} F_m(n+1) & F_m(n) \\ F_m(n) & F_m(n-1) \end{pmatrix} \tag{116}$$

where $F_m(n-1)$, $F_m(n)$, $F_m(n+1)$ are the generalized Fibonacci numbers of the order m.

Proof. We will use mathematical induction. Clearly, for n = 1,

$$G_m^1 = \begin{pmatrix} F_m(2) & F_m(1) \\ F_m(1) & F_m(0) \end{pmatrix}. \tag{117}$$

Using the seeds (21) and the recursive relation (20) we can write:

$$F_m(0) = 0, F_m(1) = 1, F_m(2) = m\, F_m(1) + F_m(0) = m. \tag{118}$$

It follows from (117) and (118) that

$$G_m^1 = \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix}. \tag{119}$$

The base of the induction is proved.

Suppose that for a given integer k our inductive hypothesis is the following:

$$G_m^k = \begin{pmatrix} F_m^{k+1} & F_m^k \\ F_m^k & F_m^{k-1} \end{pmatrix}$$

Then we can write:

$$G_m^{k+1} = G_m^k \times G_m = \begin{pmatrix} F_m(k+1) & F_m(k) \\ F_m(k) & F_m(k-1) \end{pmatrix} \times \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} mF_m(k+1) + F_m(k) & F_m(k+1) \\ mF_m(k) + F_m(k-1) & F_m(k) \end{pmatrix} = \begin{pmatrix} F_m(k+2) & F_m(k+1) \\ F_m(k+1) & F_m(k) \end{pmatrix}.$$

Theorem is proved.

The next theorem gives a formula for the determinant of the matrix (116).

Theorem 10. For a given integer n we have:

$$\text{Det } G_m^n = (-1)^n. \tag{120}$$

Proof. Using general properties of the square matrices [] we can write

$$\text{Det } G_m^n = (\text{Det } G_m)^n \tag{121}$$

Taking into consideration (115) we can write the expression (121) as follows:

$$\text{Det } G_m^n = (\text{Det } G_m)^n = (-1)^n.$$

Theorem is proved.

Theorem 11.

$$\text{Det } G_m^n = F_m(n+1) \times F_m(n-1) - F_m^2(n) = (-1)^n. \tag{122}$$

The identity (122) follows directly from the matrix (116) and Theorem 10.

Remind that the identity (122) is one of the most important identities for the generalized Fibonacci numbers of the order m. It is clear that the identity (122) is a generalization of the famous Cassini formula.

Theorem 12.

$$G_m^n = mG_m^{n-1} + G_m^{n-2} \tag{123}$$

Proof. Using the recursive relation $F_m(n+2) = mF_m(n+1) + F_m(n)$ we can represent the matrix (116) in the form:

$$G_m^n = \begin{pmatrix} mF_m(n) + F_m(n-1) & mF_m(n-1) + F_m(n-2) \\ mF_m(n-1) + F_m(n-2) & mF_m(n-2) + F_m(n-3) \end{pmatrix} =$$

$$= m\begin{pmatrix} F_m(n) & F_m(n-1) \\ F_m(n-1) & F_m(n-2) \end{pmatrix} + \begin{pmatrix} F_m(n-1) & F_m(n-2) \\ F_m(n-2) & F_m(n-3) \end{pmatrix} = mG_m^{n-1} + G_m^{n-2}$$

Theorem is proved.

Also we can represent the expression (116) in the following form:

$$G_m^{n-2} = G_m^n - mG_m^{n-1} \tag{124}$$

Basing on the recursive relations (123) and (124) we can construct the sequences of the G_m-matrices (116) for the different m. Notice that for the case m=1 the matrices G_1^n coincide with the matrices Q^n given by (15).

Consider now the case m=2. Remind that for this case a sequence of the generalized Fibonacci numbers $F_2(n)$ of the order m=2 looks as is shown in Table 4.

Table 4. A sequence of the Fibonacci numbers $F_2(n)$

n	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
m=2	169	-70	29	-12	5	-2	1	0	1	2	5	12	29	70	169

Construct now a sequence of the matrices G_2^n. For the case n=0 we will define the matrix G_m^0 as follows:

$$G_m^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (125)$$

Using the recursive relation (123) and taking into consideration the seeds (125) and (119) we can construct the matrices G_2^2, G_2^3, G_2^4 and so on as follows:

$$G_2^2 = 2\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \quad (126)$$

$$G_2^3 = 2\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 12 & 5 \\ 5 & 2 \end{pmatrix} \quad (127)$$

$$G_2^4 = 2\begin{pmatrix} 12 & 5 \\ 5 & 2 \end{pmatrix} + \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 29 & 12 \\ 12 & 5 \end{pmatrix} \quad (128)$$

Using the recursive relation (124) and taking into consideration the seeds (125) and (119) we can construct the matrices G_2^{-1}, G_2^{-2}, G_2^{-3} and so on as follows:

$$G_2^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \quad (129)$$

$$G_2^{-2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} \quad (130)$$

$$G_2^{-3} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} - 2\begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ 5 & -12 \end{pmatrix} \quad (131)$$

A sequence of the matrices G_2^n is represented in Table 5

Table 5. A sequence of the matrices G_2^n

n	0	1	2	3	4	5
G_2^n	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 12 & 5 \\ 5 & 2 \end{pmatrix}$	$\begin{pmatrix} 29 & 12 \\ 12 & 5 \end{pmatrix}$	$\begin{pmatrix} 70 & 29 \\ 29 & 12 \end{pmatrix}$
G_2^{-n}	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$	$\begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$	$\begin{pmatrix} -2 & 5 \\ 5 & -12 \end{pmatrix}$	$\begin{pmatrix} 5 & -12 \\ -12 & 29 \end{pmatrix}$	$\begin{pmatrix} -12 & 29 \\ 29 & -70 \end{pmatrix}$

Consider the case m=3. Remind that for this case a sequence of the generalized Fibonacci numbers $F_3(n)$ of the order m=3 looks as is shown in Table 6.

Table 6. A sequence of the Fibonacci numbers $F_3(n)$

n	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
m=3	-337	109	-33	10	-3	1	0	1	3	10	33	109	360

Using the seeds (125) and (118) and the recursive formulas (123) and (124) for the case m=3 we can construct the matrices G_3^n (see Table 7).

Table 7. A sequence of the matrices G_3^n

n	0	1	2	3	4	5
G_3^n	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}3 & 1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}10 & 3\\3 & 1\end{pmatrix}$	$\begin{pmatrix}33 & 10\\10 & 3\end{pmatrix}$	$\begin{pmatrix}109 & 33\\33 & 10\end{pmatrix}$	$\begin{pmatrix}360 & 109\\109 & 33\end{pmatrix}$
G_3^{-n}	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1\\1 & -3\end{pmatrix}$	$\begin{pmatrix}1 & -3\\-3 & 10\end{pmatrix}$	$\begin{pmatrix}-3 & 10\\10 & -33\end{pmatrix}$	$\begin{pmatrix}10 & -33\\-33 & 109\end{pmatrix}$	$\begin{pmatrix}-33 & 109\\109 & -360\end{pmatrix}$

It is easy to verify that all square matrices G_2^n of Table 5 and the matrices G_3^n of Table 7 posses one surprising property: all their determinants are equal +1 (for the even powers n) or -1 (for the odd powers n). In fact, the determinant of the matrix $G_2^4 = \begin{pmatrix}29 & 12\\12 & 5\end{pmatrix}$ is equal 29×5 - 12×12 = 1 and the determinant of the matrix $G_3^{-5} = \begin{pmatrix}-33 & 109\\109 & -360\end{pmatrix}$ is equal (-33)×(-360) - 109×109 = 11880 -11881 = -1.

Consider now a general case of m. Remind that the number m is a positive real number, for example, $m=\sqrt{2}$, $m=\pi$, $m=e$ (a base of natural logarithms) and so on. Using the recursive relation (123) and taking into consideration the seeds (125) and (119) we can construct the matrices G_m^2, G_m^3, G_m^4 and so on as follows:

$$G_m^2 = m\begin{pmatrix}m & 1\\1 & 0\end{pmatrix} + \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}m^2+1 & m\\m & 1\end{pmatrix} \quad (132)$$

$$G_m^3 = m\begin{pmatrix}m^2+1 & m\\m & 1\end{pmatrix} + \begin{pmatrix}m & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}m^3+2m & m^2+1\\m^2+1 & m\end{pmatrix} \quad (133)$$

$$G_m^4 = m\begin{pmatrix}m^3+2m & m^2+1\\m^2+1 & m\end{pmatrix} + \begin{pmatrix}m^2+1 & m\\m & 1\end{pmatrix} = \begin{pmatrix}m^4+3m^2 & m^3+2m\\m^3+2m & m^2+1\end{pmatrix} \quad (134)$$

Using the recursive relation (124) and taking into consideration the seeds (125) and (119) we can construct the matrices G_m^{-1}, G_m^{-2}, G_m^{-3} and so on as follows:

$$G_m^{-1} = \begin{pmatrix}m & 1\\1 & 0\end{pmatrix} - m\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & -m\end{pmatrix} \quad (135)$$

$$G_m^{-2} = \begin{pmatrix}1 & -m\\-m & m^2+1\end{pmatrix} \quad (136)$$

$$G_m^{-3} = \begin{pmatrix}-m & m^2+1\\m^2+1 & -m^3-2m\end{pmatrix} \quad (137)$$

Inverse matrices G_m^{-n}

Consider once again Table 5 and Table 7. They set the "direct" and "inverse" G_m-matrices. Comparing the "direct" (G_m^n) with the "inverse (G_m^{-n}) G_m-matrices it is easy to find a very simple method to get the "inverse" matrix G_m^{-n} from its "direct" matrix G_m^n.

In fact, if the power n of the "direct" matrix G_m^n given by (116) is even (n=2k) then for obtaining its inverse matrix G_m^{-n} it is necessary to interchange the places of the diagonal elements $F_m(n+1)$ и $F_m(n-1)$ in (116) and to take the diagonal elements $F_m(n)$ in (116) with the opposite sign. This means that for the case n=2k the "inverse" matrix G_m^{-2k} has the following form:

$$G_m^{-2k} = \begin{pmatrix} F_m(2k-1) & -F_m(2k) \\ -F_m(2k) & F_m(2k+1) \end{pmatrix} \quad (138)$$

To obtain the "inverse" matrix G_m^{-n} from the "direct" matrix G_m^n given by (116) for the case n=2k+1 it is necessary to interchange the places of the diagonal elements $F_m(n+1)$ и $F_m(n-1)$ and to take them with the opposite sign, that is:

$$G_m^{-2k-1} = \begin{pmatrix} -F_m(2k-1) & F_m(2k) \\ F_m(2k) & -F_m(2k+1) \end{pmatrix} \quad (139)$$

One more way to obtain the matrices G_m^n follows directly from the expression (116). With this aim we can represent two Fibonacci series $F_2(n+1)$ и $F_2(n)$ shifted one relative to another on the one number (Table 8).

Table 8. The shifted Fibonacci series $F_2(n+1)$ и $F_2(n)$

n	6	**5**	4	3	2	**1**	0	-1	-2	-3	-4	**-5**	-6
$F_2(n+1)$	169	**70**	**29**	12	5	**2**	**1**	0	1	-2	5	**-12**	**29**
$F_2(n)$	70	**29**	**12**	5	2	**1**	**0**	1	-2	5	-12	**29**	**-70**

If we select the number n=1 in the first row of Table 8 and then select the four Fibonacci numbers of the kind $F_2(n+1)$ and $F_2(n)$ in the lower two rows under the number 1 and to the right relative to it, then the totality of these Fibonacci numbers forms the G_m-matrix (119). The G_m-matrix is singled out by fat in Table 8. If we move in Table 8 to the left relative to the G_m-matrix, then we will get the matrices G_2^2, G_2^3, …, G_2^n, …, respectively. If we move in Table 8 to the right relative to the G_m-matrix then we will get the matrices G_2^0, G_2^{-1}, G_2^{-2}, …, G_2^{-n}, respectively. Also the Fibonacci matrices G_2^5 and the "inverse" to it Fibonacci matrix G_2^{-5} are singled out by fat in Table 8. Notice that the matrix $G_2^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is an identity matrix.

This method of obtaining the G_m^n-matrices can be used for the case of the arbitrary m.

Thus, our investigations led us to the discovery of a unique class of the square G_m-matrices given by (114) and (116). They are a wide generalization of the Hoggatt-Fibonacci Q-matrices given by (14), (15).

6. The "golden" G_m-matrices of the order m

Stakhov introduced in [26] the so-called "golden" matrices, which are a generalization of the Hoggatt-Fibonacci Q-matrix (15) for continuous domain. The above Stakhov-Fibonacci G_m-matrices of the order m given by (116) can be used for a wide generalization of Stakhov's "golden" matrices (19). We can represent the matrix (116) in the form of the two matrices given for the even (n=2k) and odd (n=2k+1) values of n:

$$G_m^{2k} = \begin{pmatrix} F_m(2k+1) & F_m(2k) \\ F_m(2k) & F_m(2k-1) \end{pmatrix} \tag{140}$$

$$G_m^{2k+1} = \begin{pmatrix} F_m(2k+2) & F_m(2k+1) \\ F_m(2k+1) & F_m(2k) \end{pmatrix} \tag{141}$$

And now we will return back again to the hyperbolic Fibonacci functions of the order m given by (71), (72). As is shown above, the generalized Fibonacci numbers of the order m are determined identically through the hyperbolic Fibonacci and Lucas functions of the order m by the correlation (75).

Using (75) we can express the matrices (140) and (141) in the terms of the hyperbolic Fibonacci functions (71) and (72) as follows:

$$G_m^{2k} = \begin{pmatrix} cF_m(2k+1) & sF_m(2k) \\ sF_m(2k) & cF_m(2k-1) \end{pmatrix} \tag{142}$$

$$G_m^{2k+1} = \begin{pmatrix} sF_m(2k+2) & cF_m(2k+1) \\ cF_m(2k+1) & sF_m(2k) \end{pmatrix} \tag{143}$$

where k is a discrete variable, k=0, ±1, ±2, ±3, … .

If we substitute the discrete variable k in the matrices (142), (143) by the continuous variable x, then we will come to the two unusual matrices that are the functions of the continuous variable x:

$$G_m^{2x} = \begin{pmatrix} cF_m(2x+1) & sF_m(2x) \\ sF_m(2x) & cF_m(2x-1) \end{pmatrix} \tag{144}$$

$$G_m^{2x+1} = \begin{pmatrix} sF_m(2x+2) & cF_m(2x+1) \\ cF_m(2x+1) & sF_m(2x) \end{pmatrix} \tag{145}$$

Notice that the "golden" matrices of the order m given by (144), (145) are a wide generalization of the "golden" matrices given by (19), which are partial cases of the matrices (144) and (145) for the case m=1, that is,

$$G_1^{2x} = Q^{2x} \text{ and } G_1^{2x+1} = Q^{2x+1} \tag{146}$$

The inverse "golden" matrices of the order m

We can represent the inverse matrices (138), (139) in the terms of the hyperbolic Fibonacci functions of the order m given by (71), (72). :

$$G_m^{-2k} = \begin{pmatrix} cF_m(2k-1) & -sF_m(2k) \\ -sF_m(2k) & cF_m(2k+1) \end{pmatrix} \tag{147}$$

$$G_m^{-2k-1} = \begin{pmatrix} -sF_m(2k) & cF_m(2k+1) \\ cF_m(2k+1) & -sF_m(2k+2) \end{pmatrix} \tag{148}$$

where k is a discrete variable, k=0, ±1, ±2, ±3, … .

If we substitute now the discrete variable k in the matrices (147), (148) by the continuous variable x, then we will come to the following matrices that are the functions of the continuous variable x:

$$G_m^{-2x} = \begin{pmatrix} cF_m(2x-1) & -sF_m(2x) \\ -sF_m(2x) & cF_m(2x+1) \end{pmatrix} \tag{149}$$

$$G_m^{-2x-1} = \begin{pmatrix} -sF_m(2x) & cF_m(2x+1) \\ cF_m(2x+1) & -sF_m(2x+2) \end{pmatrix} \quad (150)$$

Determinants of the "golden" matrices of the order m

Calculate now the determinants of the matrices (144) and (145):

$$\text{Det } G_m^{2x} = cF_m(2x+1)\times cF_m(2x-1) - [sF_m(2x)]^2 \quad (151)$$

$$\text{Det } G_m^{2x+1} = sF_m(2x+2)\times sF_m(2x) - [cF_m(2x+1)]^2 \quad (152)$$

Compare now the expression (151), (152) with the identities (99), (100) for the hyperbolic Fibonacci functions of the order *m*. Because the identities (99), (100) are valid for all values of the variable *x,* in particular, for the value 2*x,* the following identities follow from this consideration:

$$\text{Det } G_m^{2x} = 1 \quad (153)$$

$$\text{Det } G_m^{2x+1} = -1 \quad (154)$$

By analogy we can prove the following identities for the "inverse" matrices (149), (150):

$$\text{Det } G_m^{-2x} = 1 \quad (155)$$

$$\text{Det } G_m^{-2x-1} = -1 \quad (156)$$

Notice that the unusual identities (152)-(156) are a generalization of the Cassini formula for continuous domain.

Thus, our investigations, which are a continuation of our theory of the "golden" matrices [26], led us to the discovery of a unique class of the "golden" G_m-matrices given by (144) and (145). They are a wide generalization of Stakhov's "golden" matrices given by (19).

7. The improved method of the "golden" cryptography

Stakhov developed in [26] a new kind of cryptography based on the use of the "golden" matrices (19). The above "golden" direct and inverse matrices of the order *m* given by (144), (145), (149), (150) allow to improve the "golden" cryptographic method developed in [26].

Let the initial message be a "digital signal", which is any sequence of real numbers called *readings*:

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, \ldots \quad (157)$$

There are many examples of the "digital signals" of the kind (157): digital telephony, digital TV, digital measurement systems and so on.

Choose the first four readings a_1, a_2, a_3, a_4 from (157) and form from them a square 2×2-matrix *M*:

$$M = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad (158)$$

Notice that the initial matrix *M* can be considered as a *plaintext* [24].

Notice that there are 4! = 4×3×2×1 = 24 variants (permutations) to form the matrix (158) from the four readings a_1, a_2, a_3, a_4 . Designate the *i*-th permutation by P_i (i=1, 2, …, 24). The first step of cryptographic protection of the four readings a_1, a_2, a_3, a_4 is a choice of the permutation P_i .

Then we choose the direct "golden" matrices (144) or (145) as *enciphering matrices* and their inverse matrices (149), (150) as *deciphering matrices.*

Consider now the following encryption/decryption algorithms based on matrix multiplication (see Table 9).

Table 9. Encryption/decryption algorithm based on the G_m-matrices

Encryption	Decryption
$M\times G_m^{2x} = E_1(x, m)$	$E_1(x,m)\times G_m^{-2x} = M$
$M\times G_m^{2x+1} = E_2(x,m)$	$E_2(x,m)\times G_m^{-2x-1} = M$

Here M is the *plaintext* (158) that is formed according to the permutation P_i; $E_1(x, m)$, $E_2(x, m)$ are *code matrices* or *cipher texts*; G_m^{2x}, G_m^{2x+1} are the *enciphering matrices* given by (144), (145); G_m^{-2x} and G_m^{-2x-1} are the *deciphering matrices* given by (149), (150).

Notice that the encryption/decryption algorithm given by Table 9 is a partial case of the encryption/decryption algorithm developed in [26] because for the case m=1 the matrices G_1^{2x} and G_1^{2x+1} are reduced to the matrices Q^{2x} and Q^{2x+1}, respectively.

From the point of view of cryptography the main advantage of the "golden" cryptographic method given by Table 9 is an appearance of new cryptographic key, the "gnomonic" number *m,* which is a positive real number. Thus, the code matrices $E_1(x, m)$, $E_2(x, m)$ are functions of the two continuous variables x and m, what can give new possibilities for cryptographic protection.

Let us prove that the encryption/decryption algorithm given by Table 9 provides the one-valued transformation of the *plaintext M* into the *cipher text E* and then the *cipher text E* into the *plaintext M.* We will consider this transformation for the case when we choose the matrix (144) as *enciphering matrix* and the matrix (149) as *deciphering matrix.* For the given value of the cryptographic keys x and m the "golden" encryption, that is, the transformation of the plaintext M into the cipher texts $E_1(x, m)$ can be represented as follows:

$$M\times G_m^{2x} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\times\begin{pmatrix} cF_m(2x+1) & sF_m(2x) \\ sF_m(2x) & cF_m(2x-1) \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = E(x, m) \quad (159)$$

where

$$e_{11} = a_1cF_m(2x+1) + a_2sF_m(2x) \quad (160)$$
$$e_{12} = a_1sF_m(2x) + a_2cF_m(2x-1) \quad (161)$$
$$e_{21} = a_3cF_m(2x+1) + a_4sF_m(2x) \quad (162)$$
$$e_{22} = a_3sF_m(2x) + a_4cF_m(2x-1) \quad (163)$$

Consider the "golden" decryption for this case:

$$E(x,m)\times G_m^{-2x} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}\times\begin{pmatrix} cFs(2x-1) & -sFs(2x) \\ -sFs(2x) & cFs(2x+1) \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = D \quad (164)$$

where

$$d_{11} = e_{11}cF_m(2x-1) - e_{12}sF_m(2x) \quad (165)$$
$$d_{12} = - e_{11}sF_m(2x) + e_{12}cF_m(2x-1) \quad (166)$$
$$d_{21} = e_{21}cF_m(2x-1) - e_{22}sF_m(2x) \quad (167)$$
$$d_{22} = - e_{21}sF_m(2x) + e_{22}cF_m(2x-1) \quad (168)$$

For the calculation of the matrix elements given by (165)-(168) we can use the expressions (160)-(163). Then we have:

$$d_{11} = [a_1cF_m(2x+1) + a_2sF_m(2x)]\, cF_m(2x-1) - [a_1sF_m(2x_1) + a_2cF_m(2x-1)]sF_m(2x) =$$
$$= a_1cF_m(2x+1)\, cF_m(2x-1) + a_2sF_m(2x)\, cF_m(2x-1) - a_1[sF_m(2x)]^2 -$$
$$- a_2cF_m(2x-1)\, sF_m(2x) = a_1\{cF_m(2x+1)\, cF_m(2x-1) - [sF_m(2x)]^2\} \quad (169)$$

Using the fundamental identities (99), (100) we can simplify the expression (169) as follows:

$$d_{11} = a_1. \tag{170}$$

In the same manner after corresponding transformations we can write:

$$d_{12} = a_2 \tag{171}$$

$$d_{21} = a_3 \tag{172}$$

$$d_{22} = a_4 \tag{173}$$

Using (170)-(173) we can write the matrix (164) as follows:

$$D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = M \tag{174}$$

This means that a cryptographic method given by Table 9 provides the one-valid transformation of the initial plaintext M at the entrance of the coder into the same plaintext M at the exit of the decoder.

Determinants of the code matrices

Calculate now the determinant of the cipher texts, that is, the code matrices $E_1(x,m)$, $E_2(x,m)$:

$$\text{Det } E_1(x,m) = \text{Det } M \times \text{Det } G_m^{2x} \tag{175}$$

$$\text{Det } E_2(x,m) = \text{Det } M \times \text{Det } G_m^{2x+1} \tag{176}$$

If we use the identities (153), (154), we can write the expressions (175), (176) as follows:

$$\text{Det } E_1(x,m) = \text{Det } M \tag{177}$$

$$\text{Det } E_2(x,m) = -\text{ Det } M \tag{178}$$

This means that the determinants of the matrices $E_1(x,m)$ and $E_2(x,m)$ are defined identically by the determinant of the initial matrix M.

Peculiarities of the "golden" cryptography based on the G_m-matrices

As we remembered above, the "golden" cryptographic method based on the G_m-matrices (Table 9) has important distinction from Stakhov's method developed in [26]. In Stakhov's method [26] we use the only kind of the hyperbolic functions, the symmetric hyperbolic Fibonacci functions given by (16), that is, the form of the hyperbolic functions remains without change and the cryptographic protection is provided by the continuous variable x, which is used as *cryptographic key*. For the case of the "golden" cryptographic method given by Table 9 we use an infinite number of the hyperbolic Fibonacci functions of the order m given by (71), (72). At that we use the two cryptographic keys, the continuous variable x and the "gnomonic number" m, which determines the form of the used hyperbolic functions (71), (72). This means that the realization of the encryption algorithm, which is reduced to the calculation of the elements of the code matrix (159), id reduced to the fulfillment of the following calculations:

$$e_{11} = a_1 \frac{1}{\sqrt{4+m^2}}\left[\left(\frac{m+\sqrt{4+m^2}}{2}\right)^{2x+1} + \left(\frac{m+\sqrt{4+m^2}}{2}\right)^{-2x-1}\right] + $$

$$+a_2 \frac{1}{\sqrt{4+m^2}}\left[\left(\frac{m+\sqrt{4+m^2}}{2}\right)^{2x} - \left(\frac{m+\sqrt{4+m^2}}{2}\right)^{-2x}\right] \tag{179}$$

$$e_{12} = a_1 \frac{1}{\sqrt{4+m^2}} \left[\left(\frac{m+\sqrt{4+m^2}}{2} \right)^{2x} - \left(\frac{m+\sqrt{4+m^2}}{2} \right)^{-2x} \right] +$$

$$+ a_2 \frac{1}{\sqrt{4+m^2}} \left[\left(\frac{m+\sqrt{4+m^2}}{2} \right)^{2x-1} + \left(\frac{m+\sqrt{4+m^2}}{2} \right)^{-2x+1} \right] \quad (180)$$

$$e_{21} = a_3 \frac{1}{\sqrt{4+m^2}} \left[\left(\frac{m+\sqrt{4+m^2}}{2} \right)^{2x+1} + \left(\frac{m+\sqrt{4+m^2}}{2} \right)^{-2x-1} \right] +$$

$$+ a_4 \frac{1}{\sqrt{4+m^2}} \left[\left(\frac{m+\sqrt{4+m^2}}{2} \right)^{2x} - \left(\frac{m+\sqrt{4+m^2}}{2} \right)^{-2x} \right] \quad (181)$$

$$e_{22} = a_3 \frac{1}{\sqrt{4+m^2}} \left[\left(\frac{m+\sqrt{4+m^2}}{2} \right)^{2x} - \left(\frac{m+\sqrt{4+m^2}}{2} \right)^{-2x} \right] +$$

$$+ a_4 \frac{1}{\sqrt{4+m^2}} \left[\left(\frac{m+\sqrt{4+m^2}}{2} \right)^{2x-1} + \left(\frac{m+\sqrt{4+m^2}}{2} \right)^{-2x+1} \right] \quad (182)$$

The realization of the decryption algorithm, which is reduced to the calculation of the elements of the code matrix (164), means the fulfillment of the following calculations:

$$d_{11} = e_{11} \frac{1}{\sqrt{4+m^2}} \left[\left(\frac{m+\sqrt{4+m^2}}{2} \right)^{2x-1} + \left(\frac{m+\sqrt{4+m^2}}{2} \right)^{-2x+1} \right] -$$

$$- e_{12} \frac{1}{\sqrt{4+m^2}} \left[\left(\frac{m+\sqrt{4+m^2}}{2} \right)^{2x} - \left(\frac{m+\sqrt{4+m^2}}{2} \right)^{-2x} \right] \quad (183)$$

$$d_{12} = - e_{11} \frac{1}{\sqrt{4+m^2}} \left[\left(\frac{m+\sqrt{4+m^2}}{2} \right)^{2x} - \left(\frac{m+\sqrt{4+m^2}}{2} \right)^{-2x} \right] +$$

$$+ e_{12} \frac{1}{\sqrt{4+m^2}} \left[\left(\frac{m+\sqrt{4+m^2}}{2} \right)^{2x-1} + \left(\frac{m+\sqrt{4+m^2}}{2} \right)^{-2x+1} \right] \quad (184)$$

$$d_{21} = e_{21} \frac{1}{\sqrt{4+m^2}} \left[\left(\frac{m+\sqrt{4+m^2}}{2} \right)^{2x-1} + \left(\frac{m+\sqrt{4+m^2}}{2} \right)^{-2x+1} \right] +$$

$$- e_{22} \frac{1}{\sqrt{4+m^2}} \left[\left(\frac{m+\sqrt{4+m^2}}{2} \right)^{2x} - \left(\frac{m+\sqrt{4+m^2}}{2} \right)^{-2x} \right] \quad (185)$$

$$d_{22} = - e_{21}\frac{1}{\sqrt{4+m^2}}\left[\left(\frac{m+\sqrt{4+m^2}}{2}\right)^{2x} - \left(\frac{m+\sqrt{4+m^2}}{2}\right)^{-2x}\right] +$$

$$+ e_{22}\frac{1}{\sqrt{4+m^2}}\left[\left(\frac{m+\sqrt{4+m^2}}{2}\right)^{2x-1} + \left(\frac{m+\sqrt{4+m^2}}{2}\right)^{-2x+1}\right] \quad (186)$$

8. Advantages of the new method of the "golden" cryptography

Improvement of cryptographic protection

We can improve the cryptographic protection of the methods based on Table 9 if we use *multiple encryption and decryption.* This idea consists in the following. The first step of the encryption is to use the key

$$K_1=\{P_i , x_1, m_1\} \quad (187)$$

The cryptographic key (187) includes any permutation P_i of the matrix (158) and some values x_1 of the continuous variable x and the "gnomonic" number m_1 taken in random manner. As a result of the encryption we can get the code matrix the code matrix $E(P_i, x_1, m_1)$ given by (159). The second step of the encryption is to use the matrix $E(P_i, x_1, m_1)$ as the initial matrices for the new encryption. With this aim we can use the second cryptographic keys

$$K_1=\{P_j, x_2, m_2\} \quad (188)$$

where P_j is the next permutation, x_2 is the next value of x and m_2 is the next value of m.. After the fulfillment of the "golden" encryption with the key (188) we can get a new code matrix E that is a function of the two permutation P_i and P_j , the two values x_1 and x_2 and the two values m_1, m_2 , that is,

$$E = E\,(P_j , x_1, m_1;\ P_j, x_2, m_2) \quad (189)$$

In general case we can repeat this procedure k times, that is, the cryptographic key K is a totality of the k random permutations P_i , P_j, …, P_s , the k random values $x_1, x_2, ..., x_k$ and the k random values $m_1, m_2, ..., m_k$ that is,

$$K=\{P_i , x_1, m_1; P_j, x_2, m_2;\ ... ; P_s, x_k , m_k) \quad (189)$$

As the outcome of the multiple encryption, we can get the code matrix

$$E = E\,(K).$$

For the decryption we have to use the inverse cryptographic key K^{-1} that is an inverse form of the initial cryptographic key (189), that is,

$$K^{-1}=\{ P_s, x_k , m_k ; P_r, x_{k-1} , m_{k-1}; \dots ; P_j, x_2, m_2;\ P_i , x_1, m_1) \quad (190)$$

Transmission of the cryptographic keys

It is clear the "golden" cryptographic method relates to *symmetric-key cryptography.* As is well known, a problem of the key distribution is the main shortcoming of the symmetric-key cryptography. To eliminate this shortcoming, in the recent decades the so-called *public-key* or *asymmetric cryptography* was developed. In the asymmetric cryptosystems we use the two keys: *public key* and *private* or *secret key.* The encryption of the message before transmission is fulfilled by the use of the *public key* and the decryption of *cipher text* is fulfilled by the use of the *secret key*. However, the asymmetric cryptography has two shortcomings:

(1) The asymmetric cryptography uses very complicated encryption and decryption algorithms. This means that sometimes this kind of cryptography cannot be used for the protection of digital signals in real scale of time.
(2) Because the encryption and decryption algorithms are very complicated and demands complicated processors for their realization, this fact puts forward a problem to guarantee that the encryption and decryption algorithms would be fulfilled without errors (a problem of reliable computations).

To design fast and reliable cryptographic method we can join the "golden" cryptography with the asymmetric cryptography. We will use the existing asymmetric cryptosystems for the distribution of the key (189). Such approach has the following advantages:

(1) Because simplicity of the "golden" encryption/decryption algorithms, we can use the "golden" cryptographic system given by Table 9 for the fast transmission of the digital signals.
(2) We can use the unique mathematical property of the "golden" cryptography given by (177), (178) to check the encryption and decryption results.

This means that using the "golden" cryptography method given in Table 9 we can design *fast, simple for technical realization and reliable cryptosystems.*

Notice that for every session of transmission we can change the cryptographic key (189). This means that the analysis of the previous transmissions cannot be used for uncovering the current cryptographic key (189). We can change the cryptographic key (189) using a generator of random numbers. This means that we have many different ways to improve the cryptographic protection.

9. Conclusion

In author's opinion the present paper is of a great importance for the development of the contemporary theory of Fibonacci numbers and the Golden Section [1-10] and also for contemporary mathematics and computer science. In conclusion the author would like to discuss the mathematical results of this paper from this general point of view:

1. **Gazale formulas.** Since ancient time it is usual to mathematics to name new mathematical discoveries and theories by the name of the outstanding mathematicians who made these discoveries and theories (Pythagoras Theorem, Euclidean geometry, Conic sections by Apollonius, Diophantine equations, Lobachevsky's geometry, Euler's formulas, Moivre's formulas and so on). In the Fibonacci numbers theory [2, 3, 7] there are a number of the fundamental mathematical results named in the honor of the outstanding scientists. Binet formulas and Cassini formulas are the most known examples of such formulas in the Fibonacci numbers theory. Considering the Gazale formulas (51), (68) from this point of view, we should note that these formulas have great importance for the development of the contemporary Fibonacci numbers theory and go far the framework of the Fibonacci numbers theory. They generate an infinite number of the generalized Fibonacci and Lucas numbers of the order m similar to the classical Fibonacci and Lucas numbers (3), (6), which are partial cases of the new numerical sequences for the case $m=1$. Gazale formulas belong to the outstanding mathematical results and are of great importance for number theory.

2. **Hyperbolic Fibonacci and Lucas functions of the order *m*** are a wide generalization of the symmetric hyperbolic Fibonacci and Lucas functions introduced by Stakhov and Rozin in 2005 [14]. Remind that during many centuries the science, in particular, mathematics and theoretical physics, used widely the classical hyperbolic functions with the base e. These functions were used by Lobachevsky in his non-Euclidean geometry and by Minkovsky in his geometric interpretation of Einstein's theory of relativity.

Stakhov, Tkachenko and Rozin's works [13, 14] violated a monopoly of the classical hyperbolic functions in contemporary mathematics and theoretical physics. Stakhov, Tkachenko and Rozin proved that the geometry of the Living Nature (in particular, botanic phenomenon of phyllotaxis) can be modelled by the hyperbolic Fibonacci and Lucas functions with the base $\Phi_1 = \frac{1+\sqrt{5}}{2}$ (the golden ratio). It is clear that the above hyperbolic Fibonacci and Lucas functions of the order m based on the Gazale formulas extend indefinitely a number of new hyperbolic models of Nature. It is difficult to imagine, that the number of new hyperbolic functions given by formulas (71)-(74) is so much, how many exist real numbers! And all of them possess unique recursive and hyperbolic properties similar to the properties of the classical hyperbolic functions and the hyperbolic Fibonacci and Lucas functions introduced in [13, 14]. This fact is of great importance for the development of the contemporary hyperbolic geometry and theoretical physics.

3. **The "golden" matrices of the order *m*** are a wide generalization of Stakhov's "golden" matrices introduced in 2006 [26]. They are based on the hyperbolic Fibonacci and Lucas functions of the order m and are of great importance for matrix theory.
4. **A new cryptographic method based on the "golden" matrices of the order *m*** is a wide generalization of Stakhov's "golden" cryptographic method developed in 2006 [26]. This new cryptographic method improves considerably possibilities of the cryptographic protection and can lead to the design of the fast, simple for technical realization and reliable cryptosystems.

Notice that this paper is natural corollary of the preceding author's works in the "Golden Section" field [4-6, 11-28], first of all, the papers [13,14], in which Stakhov, Tkachenko and Rozin developed a new class of hyperbolic functions, and the papers [23, 25, 26], in which Stakhov developed a new class of the Fibonacci and "golden" matrices and a new kind of coding theory and cryptography.

It is clear that the Gazale formulas [9], a new class of the hyperbolic Fibonacci and Lucas functions, a new class of the "golden" matrices and a new cryptographic method developed in the present paper are of the bright examples of the "global fibonaccization" of modern science, which finds its reflection in the works of Vera W. de Spinadel [8], Jay Kappraff [9], Midhat J. Gazale [10], Mauldin and Willams [29], El Nashie [30-36], Vladimirov [37, 38], Soroko [39], Bodnar [40], Petoukhov [41] and so on.

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