Mathematics of the Golden Section: from Euclid to contemporary mathematics and computer science

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Algebra and Geometry have one and the same fate. The rather slow successes followed after the fast ones at the beginning. They left science at such step where it was still far from perfect. It happened, probably, because Mathematicians paid attention to the higher parts of the Analysis. They neglected the beginnings and did not wish to work on such field, which they finished with one time and left it behind.

Nikolay Lobachevsky

В статье приводится обзор новых результатов в области «Математики Золотого Сечения», восходящей к «Началам» Евклида (Теорема 2.11). К числу этих результатов относятся: *обобщение Теоремы Евклида о «делении в крайнем и среднем отношении», новый класс гиперболических функций - гиперболические функции Фибоначчи и Люка, обобщенные числа Фибоначчи, обобщенные золотые пропорции, обобщенный принцип золотого сечения, «золотые» алгебраические уравнения, обобщенные формулы Бине, обобщенные числа Люка, матрицы Фибоначчи, «золотые» матрицы.* Рассматриваются приложения этих математических результатов в теории чисел, теории измерения, компьютерной арифметике, теории кодирования и криптографии.

We give in the article a survey of the new results in the "Mathematics of the Golden Section", going back in its origin to Euclid's *Elements,* namely, a new class of hyperbolic functions, *a generalization of Euclid's Theorem II,11, a new class of hyperbolic functions - hyperbolic Fibonacci and Lucas functions,* the *generalized Fibonacci numbers,* the *generalized golden proportions,* the *generalized principle of the golden section,* the *golden algebraic equations,* the *generalized Binet formulas,* the *generalized Lucas numbers, Fibonacci matrices,* the *"golden" matrices*. We consider applications of these mathematical results in number theory, measurement theory, computer arithmetic, coding theory and cryptography.

Key words. The golden section, Fibonacci and Lucas numbers, Binet formulas, measurement theory, number systems, hyperbolic functions, coding theory and cryptography

1. Introduction

A division in extreme and mean ratio (DEMR) in Euclid's Elements

"The Elements" of Euclid is one of the most known mathematical work of ancient science. This scientific work was written by Euclid in the 3-d century B.C. It contains the main theories of the antique mathematics: elementary geometry, number theory, algebra, theory of proportions and ratios, methods of calculations of areas and volumes, etc. Euclid summed in this work the 300-year period of the development of the Greek mathematics and created a strong base to the further development of mathematics. During more than two millenia "The Elements" remained a basic work on the "Elementary Mathematics", which gave the origin of many fundamental theories of mathematics, in particular, geometry, number theory, measurement theory.

From "The Elements" of Euclid the following geometrical problem, which was named the problem of "Division in Extreme and Mean Ratio" (DEMR), came to us [1]. This problem was formulated in Book II of "The Elements" as follows.

Theorem II,11 (the area formulation of DEMR)**.** To divide a line *AB* into two segments, a larger one *CB* and a smaller one *AC* so that

$$S(CB) = R(AB,AC). \tag{1}$$

Remind that $S(CB)$ means the area of a square with a side *CB* and $R(AB,AC)$ means the area of a rectangle with sides *AB* and *AC.*

We can rewrite expression (1) in the following form:

$$(CB)^2 = AB\times AC \tag{2}$$

Divide now both parts of the expression (2) by *CB* and then by *AC.* Then the expression (2) takes a form of the following proportion:

$$\frac{CB}{AC} = \frac{AB}{CB}, \tag{3}$$

well-known for us as the "golden section".

We can interpret a proportion (3) geometrically: divide a line *AB* with a point *C* into two segments, a larger one *CB* and a smaller one *AC* so that a ratio of a larger segment *CB* to a smaller one *AC* is equal to a ratio of a line *AB* to a larger segment *CB.*

Designate a proportion (3) by *x.* Then, taking into consideration that $AB = AC + CB$, the proportion (3) can be written in the following form:

$$x = \frac{AB}{CB} = \frac{AC+CB}{CB} = \frac{AC}{CB} + 1 = \frac{1}{\frac{CB}{AC}} + 1 = \frac{1}{x} + 1,$$

from where the following algebraic equation follows:

$$x^2 = x + 1 \tag{4}$$

It follows from a "geometrical meaning" of the ratio (3), that the required solution of Eq. (4) has be a positive number; it follows from where that a positive root of Eq. (4) is a solution of the problem. If we designate this root by τ, then we will get:

$$\tau = \frac{1+\sqrt{5}}{2}\ . \tag{5}$$

This number is called the *golden proportion, golden mean, golden number* or *golden ratio*.

Notice that there is the following identity, which connects powers of the golden ratio:

$$\tau^n = \tau^{n-1} + \tau^{n-2} = \tau\times\tau^{n-1} \tag{6}$$

In Section 1 of Chapter 1 of the book [1] Roger Herz-Fishler analysis 84 Euclidean theorems, which, in his opinion, has relation to the DEMR, starting from Book I and ending by Book XIII. The most important of them are Theorem IV, 10 (an isosceles triangle with the angles 72° - 72° - 36°), Theorem IV, 11 (to inscribe a regular pentagon in a given circle), Theorem IV, 12 (to circumscribe a regular pentagon about a circle), Theorem IV, 13 (to inscribe a circle in a given regular pentagon), Theorem IV, 14 (to circumscribe a circle about a given regular pentagon), Theorem VI, def.3 (DEMR), Theorem VI, 30 (DEMR), Theorem XIII, 1, Theorem XIII, 4, Theorem XIII, 5, Theorem XIII, 6, Theorem XIII, 8, Theorem XIII, 9 (all about DEMR), Theorem XIII, 17 (to inscribe a dodecahedron in a sphere), Theorem XIII,18, Theorem XIII, 18 and so on. This means that DEMR goes by the "red thread" through Euclid's *Elements* and is one of the most important geometrical ideas of Euclid's *Elements.*

Why Euclid formulated Theorem II, 11? As is shown in [1] using this theorem he gave then a geometric construction of the "golden" isosceles triangle (Book IV), pentagon (Book IV), and dodecahedron (Book XIII). As is well known, the representation of the ancient Greeks about the Universe Harmony was connected with its embodiment in the Platonic Solids. Of course, that Plato's ideas about the role of the regular polyhedra in the Universe structure influenced on Euclid's *Elements*. In this famous book, which during centuries was the unique textbook on geometry, the description of the "ideal" lines and the "ideal" figures was given. A *straight line* is the most "ideal" line, and the *regular polygons* and *polyhedrons* are the most "ideal" geometric figures. It is interesting, that *The Elements* of Euclid begins from the description of *equilateral triangle*, which is the simplest regular polygon, and ends with studying of the five Platonic Solids. Notice, that a theory of the Platonic Solids are stated in the XIII, that is, final book of Euclid's *Elements*. By the way, that fact, that the theory of the Platonic Solids was placed by Euclid in the final (that is, as though the most important) book of *The Elements,* became a reason, why the ancient Greek mathematician Proclus, who was Euclid's commentator, put forward the interesting hypothesis about the true purposes of Euclid to write *The Elements*. In Proclus' opinion, Euclid wrote *The Elements* with the purpose to give a full and systematized theory of geometric construction of the "ideal" geometric figures, in particular, the five Platonic Solids, in passing he gave in *The Elements* some advanced achievements of the Greek mathematics necessary to state theory of the "ideal" geometric figures! Thus, from such unexpected point of view we can consider *The Elements* of Euclid as the first historically geometric theory of the Universe Harmony, based on the Platonic Solids and DEMR (the golden section)!

Fibonacci and Lucas numbers

Starting from ancient time "Mathematics of the Golden Section" begun to develop intensively. *Fibonacci numbers*, introduced in 1202 by the famous Italian mathematician Leonardo Pisano (by the nickname Fibonacci), were his essential contribution into the "Mathematics of the Golden Section". Fibonacci numbers are given by the following recursive relation

$$F_n = F_{n-1} + F_{n-2} \quad (7)$$

For the seeds

$$F_0 = 0; F_1 = 1 \quad (8)$$

the recursive relation (6) "generates" Fibonacci numbers:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots \quad (9)$$

In the 19th century the famous French mathematician *Lucas* introduced one more recursive numerical sequence called *Lucas numbers*:

2, 1, 3, 4, 7, 11, 18, 29, 47, … (10)

Lucas numbers (10) are given by the recursive relation

$$L_n = L_{n-1} + L_{n-2} \tag{11}$$

with the seeds

$$L_0 = 2; L_1 = 1 \tag{12}$$

Fibonacci and Lucas numbers can be extended to the side of the negative values of the index *n*. The "extended" Fibonacci and Lucas numbers are given in Table 1.

Table 1. The "extended" Fibonacci and Lucas numbers

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
F_{-n}	0	1	-1	2	-3	5	-8	13	-21	34	-55
L_n	2	1	3	4	7	11	18	29	47	76	123
L_{-n}	2	-1	3	-4	7	-11	18	-29	47	-76	123

As follows from Table 1, the terms of the "extended" numerical sequences F_n and L_n have a number of remarkable mathematical properties. For example, for the odd indexes $n = 2k+1$ the terms of the sequences F_n and F_{-n} coincide, that is, $F_{2k+1} = F_{-2k-1}$, and for the even indexes $n = 2k$ they are opposite by a sign, that is, $F_{2k} = -F_{-2k}$. For the Lucas numbers L_n it is all vise versa, that is, $L_{2k} = L_{-2k}$; $L_{2k+1} = -L_{-2k-1}$.

Binet formulas

The other famous French 19th century mathematician, *Jacques Philippe Marie Binet*, proved the following formula, which connects the golden ratio with Fibonacci and Lucas numbers:

$$\tau^n = \frac{L_n + F_n\sqrt{5}}{2}, \tag{13}$$

where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio, F_n and L_n are the "extended" Fibonacci and Lucas numbers and n takes its values from the set 0, ±1, ±2, ±3, …

Binet deduced from (13) the following formulas for the "extended" Fibonacci and Lucas numbers called *Binet formulas*:

$$F_n = \frac{\tau^n - \tau^{-n}(-1)^n}{\sqrt{5}} \tag{14}$$

$$L_n = \tau^n + \tau^{-n}(-1)^n \tag{15}$$

Q-matrix

In the second half of the 20^{th} century the interest in the golden ratio, Fibonacci and Lucas numbers increases in mathematics. A great role in the development of the “Mathematics of the Golden Section” played the brochure “Fibonacci numbers” [2] published in 1961 by the well-known Russian mathematician Nikolay Vorobyov. However, the most important events in the development of the “Mathematics of the Golden Section” were Fibonacci Association and “The Fibonacci Quarterly” created according to initiative of the famous American mathematician Verner Hoggat. In his book “Fibonacci and Lucas Numbers” [3] Verner Hoggat developed a theory of the so-called *Q-matrix*

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \quad (16)$$

In [3] the following property of the n^{th} power of the Q-matrix was proved:

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \quad (17)$$

In the form (17) the Q-matrix shows its connection to Fibonacci numbers. It is easy to prove that the determinant of the matrix (17) coincides with the famous *Cassini formula*

$$Det\ Q^n = F_{n+1}F_{n-1} - F_n^2 = (-1)^n \quad (18)$$

that was named in honor of the famous 17-th century astronomer Giovanni Cassini (1625-1712) who first derived this formula.

In the recent years “Mathematics of the Golden Section” was added by a number of the original generalizations of the classical results (1)-(18). These results are stated in author’s works [5-27]. The most important of them are the following: the *hyperbolic Fibonacci and Lucas functions* [5-8], the *generalized Fibonacci numbers* and *generalized golden sections* [9-11], the *generalized principle of the golden proportion* [13-18], the *golden algebraic equations* [19], the *generalized Binet formulas* [20], the *generalized Lucas numbers* or *Lucas p-numbers* [20], the *continuous functions for the generalized Fibonacci and Lucas numbers* [21], the *generalized Fibonacci matrices* or Q_p-*matrices* [23], the “*golden” matrices* [26] and so on. These mathematical results were used in *algorithmic measurement theory* [9-11], *codes of the golden proportion* [12], *harmony mathematics* [13, 17, 18], *ternary mirror-symmetric arithmetic* [22, 27], *a new coding theory* and *cryptography* [24-26].

The main purpose of the present article is to give a survey of the new results in the “Mathematics of the Golden Section” and its applications in mathematics and computer science.

2. Euclid’s rectangle and the “golden” brick

Euclid’s rectangle

In his Theorem II,11 Euclid described an original rectangle given by (2). Designate the length of the sides of this rectangle as follows: $AB = a$, $CB = b$ and $AC = c$. Then we can define “Euclid’s rectangle” as follows:

$$a \times c = b^2. \quad (19)$$

Taking into consideration the definition (19) we can represent Euclid’s rectangle as is shown in Fig. 1.

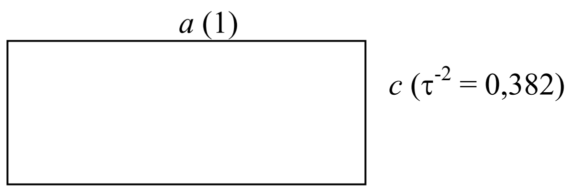

Figure 1. Euclid's rectangle

As follows from (19) and Fig.1 in Euclid's rectangle a ratio of a larger side to a smaller one is equal to a ratio of the initial line to the smaller segment in Theorem II,11; here the area of Euclid's rectangle is equal to the square of the length of the larger segment.

If we choose a unit segment as an initial line in Theorem II,11 (a=1), then the lengths of the larger and smaller segments in Theorem II,11 will be proper fractions always and then we can write the expression (19) as follows:

$$c = b^2. \quad (20)$$

It follows from (20) the following formulation of Theorem II,11 for the unit segment.

Theorem II,11 (for the unit segment). Divide a unit segment into two unequal segments in such proportion that a length of the smaller segment was equal to the square of the length of the larger segment.

If the initial segment in Theorem II,11 is equal 1 (a = 1), then the lengths of the larger and smaller segments in Theorem II,11 are equal $b = \tau^{-1}$ and $c = \tau^{-2}$ respectively. This means that Euclid's rectangle is a peculiar "golden" rectangle with the side ratio τ^2. It follows from this consideration that Euclid in his Theorem II,11 not only formulated a problem of the DEMR but also discovered a new kind of the "golden" rectangle with the side ratio $\boldsymbol{\tau^2}$ (Fig.1).

Notice that for the case of the unit segment the identity (6) takes the following form:

$$1 = \tau^{-1} + \tau^{-2} = 0{,}618 + 0{,}382 \quad (21)$$

The identity (21) expresses the famous "Principle of the Golden Section" well-known for us from the ancient culture.

The "golden" brick

Consider now "Euclid's rectangle" with the larger side equal τ (the golden ratio), and the smaller side equal τ^{-1} (Fig.2). Draw a diagonal DB in "Euclid's rectangle in Fig.2.

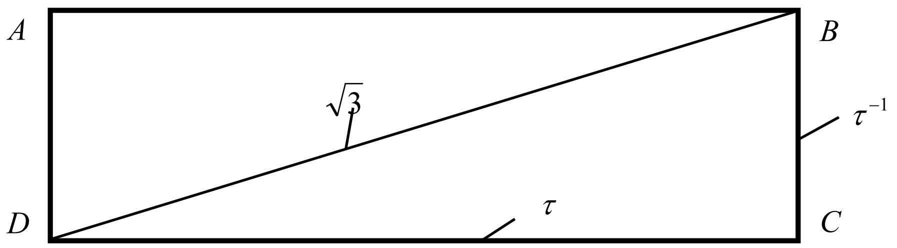

Figure 2. A diagonal of "Euclid's rectangle"

Using Pythagoras Theorem we can write:

$$DB^2 = DC^2 + BC^2 = \tau^2 + \tau^{-2} \quad (22)$$

According to Binet formula (15) $\tau^2 + \tau^{-2} = 3$ and $DB^2 = 3$, from which follows $DB = \sqrt{3}$.

"Euclid's rectangle" in Fig.2 together with the classical "golden" rectangle with the side ratio equal to the golden ratio τ can be used for the design of a peculiar rectangular parallelepiped called *"golden" brick* (Fig. 3).

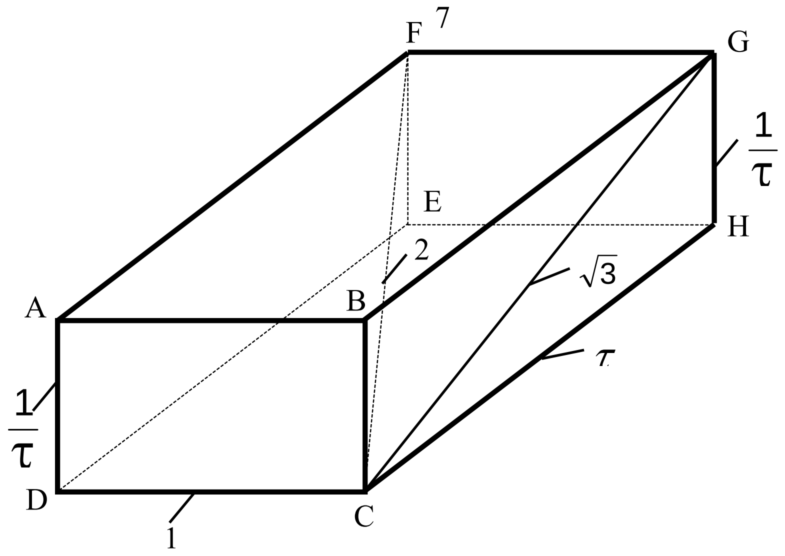

Figure 3. The "golden" brick

The faces of the "golden" brick in Fig.3 are peculiar "golden" rectangles. In particular, the face *ABCD* is a classical "golden" rectangle with the side ratio $AB:BC = 1 : \tau^{-1} = \tau$, the face *ABGF* is a classical "golden" rectangle with the side ratio $AF:AB = \tau : 1 = \tau$, at last, the face *BCHG* is Euclid's rectangle with the side ratio $BG:BC = \tau : \tau^{-1} = \tau^2$.

Using Pythagoras Theorem we can calculate a diagonal *CF* of the "golden" brick:

$$CF = \sqrt{FG^2 + CG^2} = \sqrt{1 + (\sqrt{3})^2} = 2\,.$$

Notice that the "golden" brick was used widely as a shape of the building blocks in the Gothic temples. There is hypothesis that a surprising strength of the Gothic temples is connected with the use of the "golden" bricks.

3. A generalization of Euclid's DEMR

A problem of the DEMR given by TheoremII,11 allows the following generalization. Let us give the integer non-negative number *p* (*p*=0, 1, 2, 3, …) and divide a line *AB* by a point *C* in the following ratio (Fig. 4):

$$\frac{CB}{AC} = \left(\frac{AB}{CB}\right)^p\,. \tag{23}$$

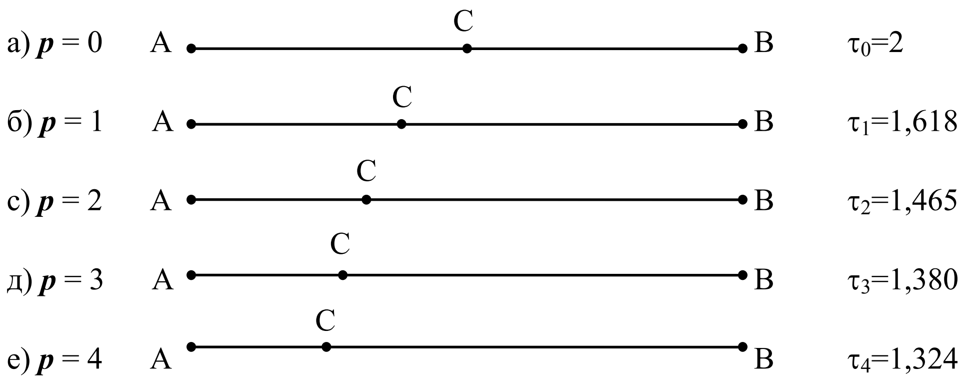

Figure 4. The golden p-sections (p = 0, 1, 2, 3, ...).

Designate by x a ratio $AB:CB = x$; then according to (23) a ratio $CB:AC = x^p$. On the other hand, $AB = AC + CB$, from where the following algebraic equation follows:

$$x^{p+1} = x^p + 1. \quad (24)$$

Designate by τ_p a positive root of the algebraic equation (24).

Eq. (24) describes an infinite number of divisions of the line segment AB in the ratio (23) because every p "generates" its own variant of the division (23). For the case p=0 we have: $\tau_p = 2$ and then the division (23) is reduced to the classical dichotomy (Fig. 4-a). For the case p=1 we have: $\tau_p = \tau = \frac{1+\sqrt{5}}{2}$ (the golden ratio) and the division (23) coincides with the classical golden section or DEMR (Fig. 4-b). This fact is a cause why the divisions of the line segment in the ratio (23) were called the *generalized golden sections* or the *golden p-sections* [9] and the positive roots τ_p of the algebraic equation (25) were called the *generalized golden proportions* or the *golden p-proportions* [9].

Notice that there is a fundamental distinction between the division of the line segment in Fig. 4-a and the rest divisions in Fig. 4-b, c, d, e from the point of view "symmetry" and "asymmetry". The division in Fig. 4-a is based on the "Dichotomy Principle" and reflects the "Symmetry Principle". The divisions in Fig. 4-b, c, d, e are "asymmetric" divisions and reflect the "Asymmetry Principle".

It follows from the equation (24) the following fundamental identity that connects the adjacent powers of the golden *p*-proportion τ_p:

$$\tau_p^n = \tau_p^{n-1} + \tau_p^{n-p-1} = \tau_p \times \tau_p^{n-1}. \quad (25)$$

where n=0, ±1, ±2, ±3, … .

Notice that for the case p=1 the general identity (25) is reduced to the identity (6).

4. A generalization of Euclid's theorem II,11

As mentioned above, the golden *p*-sections given by the proportion (23) is a generalization of the classical DEMR given by (3). However, in Euclid's Theorem II,11 the DEMR is formulated in the form (2). We can try to formulate the proportion (23) in the form (2). Start from a partial case p=2. For this case the proportion (23) takes the following form:

$$\frac{AC}{CB} = \left(\frac{AB}{AC}\right)^2 \quad (26)$$

Designate the lengths of the segments *AB, AC* и *CB* in (26) as follows: $AB = a$, $AC = b$ and $CB = c$. Then the proportion (26) can be represented in the form:

$$a^2 \times c = b^3. \quad (27)$$

We can give the expression (27) the following geometric interpretation. The right part of the equality (27) can be interpreted as a volume of the cube with the side equal to *b,* that is, to the length of the larger segment *AC,* arising at the division of a line segment *AB* in the golden 2-proportion (26). The left part of the equality (27) can be interpreted as a volume of the rectangular parallelepiped. This parallelepiped has in its basis a square with the side, which is equal to *a,* that is, to the length of the initial segment *AB,* and its height is equal to *c*, that is, to the length of the smaller segment *CB* in the proportion (26).

Then, taking into consideration (26) и (27), we can formulate a new geometric problem about division of a line in the golden 2-proportion, which is a generalization of Euclid's DEMR.

A generalization of DEMR (a division in the golden 2-proportion). Divide a given line *AB* into two segments, a smaller one *AC* and a larger one *CB* so that a volume of a rectangular parallelepiped, which basis is a square with the side equal to the initial line *AB* and which height is equal to the larger segment *CB,* was equal to a volume of a cube with a side equal to the larger segment *AC.*

A rectangular parallelepiped appeared in this problem consists of the 6 faces. The top and bottom faces are squares with the sides equal to length of the initial segment *a*; the lateral faces are rectangular rectangles with the sides equal to *a* and *c*. This rectangles are similar to Euclid's rectangle in Fig. 1 with that distinction that a ratio of its side *a:c* for the given case is equal to the square of the golden 2-proportion τ_2, that is,

$$\frac{a}{c} = \tau_2^2 \quad (28)$$

We will name indicated geometric figure *Euclid's rectangular parallelepiped.* Thus, according to (28) in Euclid's rectangular parallelepiped a ratio of the side of its basis to its height is equal to the square of the golden 2-proportion τ_2; here according to (27) its volume is equal the length of a larger segment in the proportion (26).

If the initial segment *AB* is a unite segment (*AB*=1), then the equality (27) takes the following form:

$$c = b^3. \tag{29}$$

And then we can formulate the following geometric problem of a division of the unite segment in the golden 2-proportion.

A problem of division of the unite segment in the golden 2-proportion. Divide a unite segment into two unequal segments in such proportion that the length of the smaller segment is equal to the cube of the length of the larger segment.

Notice that the formulated problem expresses the following property of the golden 2-proportion:

$$1 = \tau_2^{-1} + \tau_2^{-3} = 0{,}6823 + 0{,}3177. \tag{30}$$

For a general case of p a proportion (23) can be represented in the following form:

$$a^p \times c = b^{p+1}. \tag{31}$$

Using geometric language we can interpret the equality (31) as follows. The right part of the equality (31) is a volume of a hypercube in the (p+1)-dimensional space with the side equal to the length b of the larger segment in the problems of division of a line in the golden p-proportion. The left part of the equality (31) is a volume of Euclid's hypercube in the (p+1)-dimensional space, in which the p sides are equal to the length a of the initial segment in the problem of division of a line in the golden p-proportion, and the (p+1)-th side (its "height") is equal to the length c of the larger segment in the problem of division of a line in the golden p-proportion.

In essence, a formulated problem is a generalization of Euclid's DEMR.

At least, consider this problem for the case of the unite segment (a=1). Then the equality (31) takes the following form:

$$c = b^{p+1}. \tag{32}$$

Taking into consideration (32) we can formulate the following problem.

A problem of division of the unite segment in the golden *p*-proportion. For a given p=0, 1, 2, 3, ... divide a unite segment into two unequal segments in such proportion that the length of the smaller segment is equal to the (p+1)-th degree of the length of the larger segment.

5. The generalized principle of the golden section

And now we divide all terms of the identity (25) by τ_p^n. The following identity follows as a result of this division:

$$1 = \tau_p^0 = \tau_p^{-1} + \tau_p^{-p-1}. \tag{33}$$

Using (25) and (33) we can construct the following "dynamic" model of the "Unit" decomposition using the golden p-proportion:

$$1 = \tau_p^0 = \tau_p^{-1} + \tau_p^{-(p+1)}$$
$$\tau_p^{-(p+1)} = \tau_p^{-(p+1)-1} + \tau_p^{-2(p+1)}$$
$$\tau_p^{-2(p+1)} = \tau_p^{-2(p+1)-1} + \tau_p^{-3(p+1)} \qquad (34)$$

$$1 = \tau^0 = \tau_p^{-1} + \tau_p^{-(p+1)-1} + \tau_p^{-2(p+1)-1} + \tau_p^{-3(p+1)-1} + \ldots = \sum_{i=1}^{\infty} \tau_p^{-(i-1)(p+1)-1}$$

The main result of the above consideration is finding a more general principle of the "Unit" division given by the following identity:

$$1 = \tau_p^{-1} + \tau_p^{-(p+1)} = \sum_{i=1}^{\infty} \tau_p^{-(i-1)(p+1)-1}, \qquad (35)$$

where τ_p is the golden p-proportion, $p \in \{0, 1, 2, 3, \ldots\}$.

Notice that the equality (35) includes in itself many useful principles. For example, for the case $p=0$ we have $\tau_p = \tau_0 = 2$ and the equality (33) takes the following form:

$$1 = 2^{-1} + 2^{-1} \qquad (36)$$

The equality (36) expresses so-called "*Dichotomy Principle*", which came to us from antiquity.

For the case $p=1$ the golden p-proportion τ_p is reduced to the classical golden proportion τ and the equality (33) takes the following form:

$$1 = \tau^{-1} + \tau^{-2} \qquad (37)$$

The equality (37) expresses so-called *"Golden Section Principle"*, which came to us from antiquity too.

Table 2 represents the generalized principle of the golden section in analytical and numerical form.

Table 2. The generalized principle of the golden section

p	**Analytical expression** $1 = \tau_p^{-1} + \tau_p^{-(p+1)} = \sum_{i=1}^{\infty} \tau_p^{-(i-1)(p+1)-1}$	**Numerical expression**
0	$1 = \tau_0^{-1} + \tau_0^{-1}$	**$1 = 2^{-1} + 2^{-1}$**
1	$1 = \tau^{-1} + \tau^{-2}$	**1 = 0,6180 + 0,3820**
2	$1 = \tau_2^{-1} + \tau_2^{-3}$	**1 = 0,6823 + 0,3177**
3	$1 = \tau_3^{-1} + \tau_3^{-4}$	**1 = 0,7245 + 0,2755**
4	$1 = \tau_4^{-1} + \tau_4^{-5}$	**1 = 0,7549 + 0,2451**

5	$1=\tau_5^{-1}+\tau_5^{-6}$	**1 = 0,7781 + 0,2219**
6	$1=\tau_6^{-1}+\tau_6^{-7}$	**1 = 0,7965 + 0,1883**

6. Fibonacci *p*-numbers

Represent now Pascal triangle in the form of the following numerical Table 3

Table 3. Pascal triangle

1	1	1	1	1	1	1	1	1	1
	1	2	3	4	5	6	7	8	9
		1	3	6	10	15	21	28	36
			1	4	10	20	35	56	84
				1	5	15	35	70	126
					1	6	21	56	126
						1	7	28	84
							1	8	36
								1	9
									1
1	2	4	8	16	32	64	128	256	512

Shift now every row of Pascal triangle by one column to the right with respect to the preceding column and consider the *"deformed" Pascal triangle* (Table 4).

Table 4. "Deformed" Pascal triangle

1	1	1	1	1	1	1	1	1	1	1	1
		1	2	3	4	5	6	7	8	9	10
				1	3	6	10	15	21	28	36
						1	4	10	20	35	56
								1	5	15	35
										1	6
1	1	2	3	5	8	13	21	34	55	89	144

If we sum the binomial coefficients of the "deformed" Pascal triangle we will come unexpectedly to the Fibonacci numbers (9)!

If we shift every row of the initial Pascal triangle by p columns (p=0, 1, 2, 3, …) to the right with respect to the preceding column and then sum the binomial coefficients of the new "deformed" Pascal triangle by columns we will come to the numerical sequence that is expressed by the following recurrence relation:

$$F_p(n) = F_p(n-1)+F_p(n-p-1) \quad \text{for} \quad n>p+1; \tag{38}$$

$$F_p(1) = F_p(2) = ... = F_p(p+1) = 1. \tag{39}$$

Notice that the recursive relation (38) with the seeds (39) gives an infinite number of new numerical sequences. Moreover, the "binary" sequence 1, 2, 4, 8, 16, …, is special case of this sequence for p=0 and the classical Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, … is special case of this sequence for p=1! We will name the numerical sequence generated by (38), (39) *Fibonacci p-numbers* [9].

It is proved in [9] that Fibonacci p- numbers can be expressed in terms of the binomial coefficients as follows:

$$F_p(n+1) = C_n^0 + C_{n-p}^1 + C_{n-2p}^2 + C_{n-3p}^3 + C_{n-4p}^4 + \ldots \quad (40)$$

For the case p=0 Eq. (40) takes the following form:

$$C_n^0 + C_n^1 + \ldots + C_n^n = 2^n \quad (41)$$

widely used in combinatorial analysis.

For p=1 we have:

$$F_1(n+1) = C_n^0 + C_{n-1}^1 + C_{n-2}^2 + C_{n-3}^3 + C_{n-4}^4 + \ldots, \quad (42)$$

This formula connects Fibonacci numbers to the binomial coefficients.

If we consider a ratio of two adjacent Fibonacci p-numbers $F_p(n+1)/F_p(n)$ and direct n to infinity, we will get the following result:

$$\lim_{n\to\infty} \frac{F_p(n+1)}{F_p(n)} = \tau_p \quad (43)$$

where τ_p a positive root of Eq. (24).

This means that the golden p-proportions τ_p (p=0, 1, 2, 3, …) express certain deep, unknown until now properties of Pascal triangle. **Thus, the above considerations show a deep connection of Pascal triangle to Euclid's DEMR. This connection is expressed through the golden p-proportions τ_p, which are positive roots of the algebraic equation (24).**

7. Properties of the "golden" algebraic equations

We will name Eq. (24) *golden algebraic equation* [19]. Eq. (24) is algebraic equation of the (p+1)-th degree and therefore has p+1 roots: $x_1, x_2, x_3, \ldots, x_p, x_{p+1}$. We will consider that the root x_1 always coincides with the golden p-proportion τ_p, i.e., $x_1 = \tau_p$. It follows from Eq. (24) the following identity for the roots x_k:

$$x_k^n = x_k^{n-1} + x_k^{n-p-1} = x_k \times x_k^{n-1} \quad (44)$$

.

It is proved in [19] that the roots of Eq. (24) satisfy the following identities:

$$x_1+x_2+x_3+x_4+\ldots+x_p+x_{p+1} = 1 \quad (45)$$

$$x_1x_2x_3x_4\ldots x_{p-1}x_px_{p+1} = (-1)^p \quad (46)$$

It is proved in [19] that for p=1, 2, 3, ... and k=1, 2, 3, …, p we have the following identity for the roots of Eq. (24):

$$x_1^k + x_2^k + x_3^k + x_4 + \ldots + x_p^k + x_{p+1}^k = 1. \quad (47)$$

Also it is proved in [19] that there are an infinite number of the "golden" algebraic equations having a general root τ_p:

$$x^n = F_p(n-p+1)x^p + \sum_{t=0}^{p-1}\left[F_p(n-p-t)x^t\right] \quad (48)$$

where $n \geq p+1$.

In particular, for the case p=1, Eq. (48) is reduced to the following:

$$x^n = F_n x + F_{n-1}, \quad (49)$$

where F_n, F_{n-1} are Fibonacci numbers.

For the case n=4 Eq. (49) takes the form:

$$x^4 = 3x + 2. \quad (50)$$

Eq. (50) leads us to the unexpected result. It describes the energy condition of the molecule butadiene, a chemical used for rubber production. The famous American physicist and Laureate of

Nobel Prize Richard Feynman expressed his admiration for Eq. (50) in the following words: *"What miracles exist in mathematics! According to my theory, the golden proportion of the ancient Greeks gives the condition for the minimal energy of the butadiene molecule."*

This connection to chemistry raises our level of interest in the golden algebraic equations of the kind (50). Perhaps these equations could be used to model the energy conditions of other molecules.

8. The generalized Binet formulas for the Fibonacci and Lucas *p*-numbers

Binet formulas for the Fibonacci p-numbers

It is proved in [20] that for a given integer $p>0$ any Fibonacci p-number $F_p(n)$ for integer n can be represented in terms of the roots of the golden algebraic Eq. (19) in the following form:

$$F_p(n) = k_1(x_1)^n + k_2(x_2)^n + \dots + k_{p+1}(x_{p+1})^n \tag{51}$$

where $k_1, k_2, \dots, k_{p+1}$ are constant coefficients that are solutions of the following system of algebraic equations:

$$\begin{aligned} F_p(0) &= k_1 + k_2 + \dots + k_{p+1}=0 \\ F_p(1) &= k_1x_1 + k_2x_2 + \dots + k_{p+1}x_{p+1}=1 \\ F_p(2) &= k_1(x_1)^2 + k_2(x_2)^2 + \dots + k_{p+1}(x_{p+1})^2=1 \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ F_p(p) &= k_1(x_1)^p + k_2(x_2)^p + \dots + k_{p+1}(x_{p+1})^p=1 \end{aligned} \tag{52}$$

In particular, for $p=1$ the Fibonacci p-numbers are reduced to the classical Fibonacci numbers F_n. For this special case the expression (51) is reduced to the following well-known expression [3,4]:

$$F_n = k_1(x_1)^n + k_2(x_2)^n, \tag{53}$$

where x_1, x_2 are the roots of Eq. (4):

$$x_1 = \tau = \frac{1+\sqrt{5}}{2};\ x_2 = -\frac{1}{\tau}, \tag{54}$$

and coefficients k_1, k_2 are solutions of the system of equations:

$$\begin{cases} F_0 = k_1 + k_2 = 0 \\ F_1) = k_1\tau + k_2(-1/\tau) = 1 \end{cases} \tag{55}$$

where,

$$k_1 = \frac{1}{\sqrt{5}};\ k_2 = -\frac{1}{\sqrt{5}}. \tag{56}$$

If we substitute (54) and (56) into Eq. (53), we will get Binet formula for Fibonacci numbers in the following form:

$$F_n = \frac{\tau^n - \left(-\frac{1}{\tau}\right)^n}{\sqrt{5}}. \tag{57}$$

Binet formulas for the Lucas p-numbers

It is known [3,4] that the classical Lucas numbers L_n can be expressed in terms of the roots x_1, x_2 of Eq. (4) in the following form:

$$L_n = x_1^n + x_2^n = \tau^n + \left(-\frac{1}{\tau}\right)^n \quad (58)$$

It is proved in [20] that the sum of the n-th powers of the roots $x_1, x_2, \ldots, x_{p+1}$ of the Eq. (24) determines a new class of recursive sequences referred as *Lucas p-numbers* $L_p(n)$, where p=1, 2, 3, ..., i.e.,

$$L_p(n) = (x_1)^n + (x_2)^n + \ldots + (x_{p+1})^n \quad (59)$$

Using properties (44)-(46) of Eq. (24), we can find from (59) the following recursive relation:

$$L_p(n) = L_p(n-1) + L_p(n-p-1) \quad (60)$$

with the seeds:

$$L_p(0) = p+1 \quad (61)$$

$$L_p(1) = L_p(2) = ... = L_p(p) = 1. \quad (62)$$

Table 5 gives the values of the Lucas p-numbers for the cases p=1, 2, 3, 4.

Table 5. Lucas *p*-numbers

N	0	1	2	3	4	5	6	7	8	9	10	11	12
$L_1(n)$	2	1	3	4	7	11	18	29	47	76	123	199	322
$L_2(n)$	3	1	1	4	5	6	10	15	21	31	46	67	98
$L_3(n)$	4	1	1	1	5	6	7	8	13	19	26	34	47
$L_4(n)$	5	1	1	1	1	6	7	8	9	10	16	23	31

Thus, the main result of reference [20] is a generalization of the Binet formulas and the development of a new class of recursive numerical sequences, the *generalized Lucas numbers* or *Lucas p-numbers,* given either in analytical form (59) or in recursive form (60)-(62). These new recursive numerical sequences are of great theoretical interest, and they may prove useful for modeling certain natural processes.

9. Hyperbolic Fibonacci and Lucas functions

Stakhov and Tkachenko's approach

We can write Binet formulas for Fibonacci and Lucas numbers as follows:

$$L_n = \begin{cases} \tau^{2k} + \tau^{-2k} \\ \tau^{2k+1} - \tau^{-(2k+1)} \end{cases} \quad (63)$$

$$F_n = \begin{cases} \dfrac{\tau^{2k+1} + \tau^{-(2k+1)}}{\sqrt{5}} \\ \dfrac{\tau^{2k} - \tau^{-2k}}{\sqrt{5}} \end{cases} \quad (64)$$

where the discrete variable k takes its values from the set 0, ±1, ±2, ±3, … .

If we compare Binet formulas (63) and (64) to the classical hyperbolic functions

$$shx = \frac{e^x - e^{-x}}{2}, \quad (65)$$

$$chx = \frac{e^x + e^{-x}}{2}. \quad (66)$$

we can see a similarity between them.

In [5], the discrete variable k in formulas (63) and (64) was substituted by the continuous variable x that takes its values from the set of real numbers. Consequently, the following continuous functions, which are called *the hyperbolic Fibonacci and Lucas functions,* were introduced:

The hyperbolic Fibonacci sine

$$sF(x) = \frac{\tau^{2x} - \tau^{-2x}}{\sqrt{5}} \tag{67}$$

The hyperbolic Fibonacci cosine

$$cF(x) = \frac{\tau^{2x+1} + \tau^{-(2x+1)}}{\sqrt{5}} \tag{68}$$

The hyperbolic Lucas sine

$$sL(x) = \tau^{2x+1} - \tau^{-(2x+1)} \tag{69}$$

The hyperbolic Lucas cosine

$$cL(x) = \tau^{2x} + \tau^{-2x} \tag{70}$$

The connection between Fibonacci (F_n) and Lucas (L_n) numbers and the hyperbolic Fibonacci and Lucas functions (67)-(70) is given by the following identities:

$$sF(k) = F_{2k}; \quad cF(k) = F_{2k+1}; \quad sL(k) = L_{2k+1}; \quad cL(k) = L_{2k}, \tag{71}$$

where k is an arbitrary integer.

A symmetrical representation of the hyperbolic Fibonacci and Lucas functions
(Stakhov and Rozin's approach)

Compare now the hyperbolic Fibonacci and Lucas functions to the classical hyperbolic functions. It is easy to see that, in contrast to the classical hyperbolic functions, the graph of the Fibonacci cosine is asymmetric, while the graph of the Lucas sine is asymmetric relative to the origin of the coordinates. This restricts applications of the new class of hyperbolic functions given by (67)-(70).

Based on an analogy between Binet formulas (14) and (15) and the classical hyperbolic functions (65) and (66) Stakhov and Rozin introduced in [7] so-called *symmetric hyperbolic Fibonacci and Lucas functions*:

Symmetric hyperbolic Fibonacci sine

$$sFs(x) = \frac{\tau^{x} - \tau^{-x}}{\sqrt{5}} \tag{72}$$

Symmetric hyperbolic Fibonacci cosine

$$cFs(x) = \frac{\tau^{x} + \tau^{-x}}{\sqrt{5}} \tag{73}$$

Symmetric hyperbolic Lucas sine

$$sLs(x) = \tau^{x} - \tau^{-x} \tag{74}$$

Symmetric Lucas cosine

$$cLs(x) = \tau^{x} + \tau^{-x} \tag{75}$$

Fibonacci and Lucas numbers are determined from the symmetric Fibonacci and Lucas functions by the following substitutions,

$$F_n = \begin{cases} sFs(n), & for \quad n = 2k \\ cFs(n), & for \quad n = 2k+1 \end{cases}; \qquad L_n = \begin{cases} cLs(n), & for \quad n = 2k \\ sLs(n), & for \quad n = 2k+1 \end{cases}. \tag{76}$$

It is easy to construct graphs of the symmetric Fibonacci and Lucas functions (Fig. 5, 6).

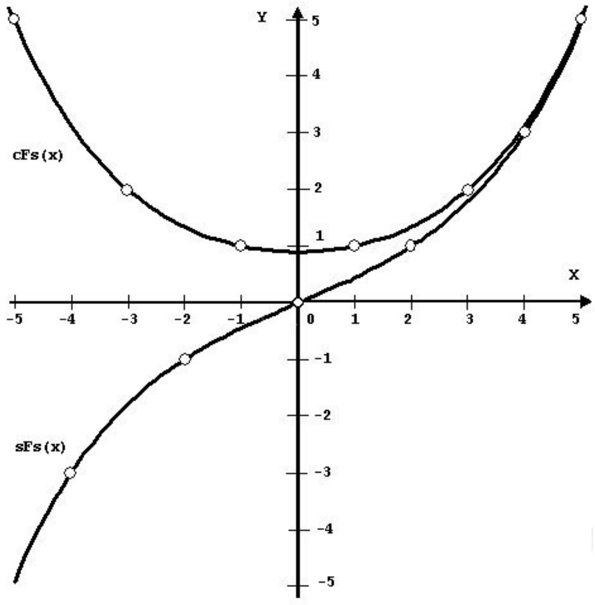

Figure 5. The symmetric Fibonacci functions

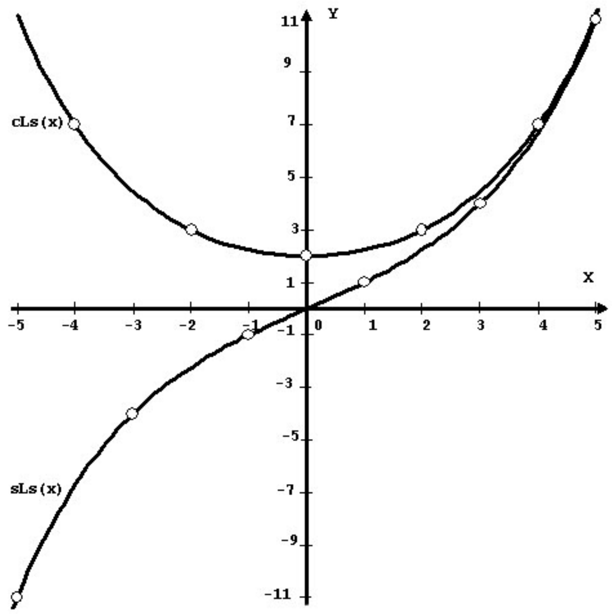

Figure 6. The symmetric Lucas functions

Their graphs have a symmetric form and are similar to the graphs of the classical hyperbolic functions. However, at the point x=0, the symmetric Fibonacci cosine $cFs(x)$ takes the value $cFs(0)=\frac{2}{\sqrt{5}}$, while the symmetric Lucas cosine $cLs(x)$ takes the value $cLs(0)$ = 2. It is also important to note that the Fibonacci numbers F_n , with even indices, are values of the symmetrical Fibonacci sine, $sFs(x)$, and the Fibonacci numbers, with odd indices, are values of the symmetric Fibonacci cosine, $cFs(x)$. On the other hand, the Lucas numbers, with the even indices, are values of the symmetrical Lucas cosine $cLs(x)$ and the Lucas numbers, with the odd indices, are values of the symmetrical Lucas cosine $sLs(x)$.

The symmetric hyperbolic Fibonacci and Lucas functions are related to the classical hyperbolic functions by the identities:

$$sFs(x)=\frac{2}{\sqrt{5}}sh(\ln(\alpha)\cdot x); \qquad cFs(x)=\frac{2}{\sqrt{5}}ch(\ln(\alpha)\cdot x);$$

$$sLs(x)\;=\;2\;sh(\ln(\alpha)\cdot x); \qquad cLs(x)\;=\;2\;ch(\ln(\alpha)\cdot x).$$

The symmetric hyperbolic Fibonacci and Lucas functions are related among each other by the identities:

$$sFs(x)\;=\;\frac{sLs(x)}{\sqrt{5}}\quad;\quad cFs(x)\;=\;\frac{cLs(x)}{\sqrt{5}}\quad.$$

The recursive properties of the symmetric hyperbolic Fibonacci and Lucas functions

The symmetric hyperbolic Fibonacci and Lucas functions (72)-(75) are generalizations of Fibonacci and Lucas numbers, and therefore they have *recursive properties*. On the other hand, they are similar to the classical hyperbolic functions, and therefore they have *hyperbolic properties.*

Table 6 lists the well known identities for Fibonacci and Lucas numbers and the corresponding identities for the symmetrical hyperbolic Fibonacci and Lucas functions.

Table 6. The recursive properties of the symmetrical hyperbolic Fibonacci and Lucas functions

The identities for Fibonacci and Lucas numbers	The identities for the symmetric hyperbolic Fibonacci and Lucas functions	
$F_{n+2} = F_{n+1} + F_n$	$sFs(x+2) = cFs(x+1) + sFs(x)$	$cFs(x+2) = sFs(x+1) + cFs(x)$
$F_n^{\ 2} - F_{n+1} F_{n-1} = (-1)^{n+1}$	$[sFs(x)]^2 - cFs(x+1)\ cFs(x-1) = -1$	$[cFs(x)]^2 - sFs(x+1)\ sFs(x-1) = 1$
$L_{n+2} = L_{n+1} + L_n$	$sLs(x+2) = cLs(x+1) + sLs(x)$	$cLs(x+2) = sLs(x+1) + cLs(x)$
$L_n^2 - 2(-1)^n = L_{2n}$	$[sLs(x)]^2 + 2 = cLs(2x)$	$[cLs(x)]^2 - 2 = cLs(2x)$
$F_{n+1} + F_{n-1} = L_n$	$cFs(x+1) + cFs(x-1) = cLs(x)$	$sFs(x+1) + sFs(x-1) = sLs(x)$
$F_n + L_n = 2F_{n+1}$	$cFs(x) + sLs(x) = 2sFs(x+1)$	$sFs(x) + cLs(x) = 2cFs(x+1)$

For example, the famous Cassini formula

$$F_n^{\ 2} - F_{n+1} F_{n-1} = (-1)^{n+1}, \quad (77)$$

which is an important identity connecting three adjacent Fibonacci numbers, can be generalized to the symmetric hyperbolic Fibonacci functions as,

$$[sFs(x)]^2 - cFs(x+1)\ cFs(x-1) = -1 \quad (78)$$

$$[cFs(x)]^2 - sFs(x+1)\ sFs(x-1) = 1, \quad (79)$$

It is clear that the identities (78), (79) can be considered to be generalizations of the Cassini formula to continuous functions.

The hyperbolic properties of the symmetric hyperbolic Fibonacci and Lucas functions

Some of the hyperbolic properties of the symmetric hyperbolic Fibonacci and Lucas functions have analogues to the properties of classical hyperbolic functions as shown in Table 7.

Table 7. The hyperbolic properties of the symmetric hyperbolic Fibonacci and Lucas functions

Classical hyperbolic function	Symmetric hyperbolic Fibonacci function	Symmetric hyperbolic Lucas function
$[ch(x)]^2 - [sh(x)]^2 = 1$	$[cFs(x)]^2 - [sFs(x)]^2 = \frac{4}{5}$	$[cLs(x)]^2 - [sLs(x)]^2 = 4$
$ch(x\pm y) = ch(x)ch(y) \pm$ $\pm sh(x)sh(y)$	$\frac{2}{\sqrt{5}} cFs(x\pm y) = cFs(x)cFs(y) \pm$ $\pm sFs(x)sFs(y)$	$2cLs(x\pm y) = cLs(x)cLs(y) \pm$ $\pm sLs(x)sLs(y)$
$sh(x\pm y) = sh(x)ch(y) \pm ch(x)sh(y)$	$\frac{2}{\sqrt{5}} sFs(x\pm y) = sFs(x)cFs(y) \pm$ $\pm cFs(x)sFs(y)$	$2sLs(x\pm y) = sLs(x)cLs(y) \pm$ $\pm cLs(x)sLs(y)$
$ch(2x) = [ch(x)]^2 + [sh(x)]^2$	$\frac{2}{\sqrt{5}} cFs(2x) = [cFs(x)]^2 +$ $+[sFs(x)]^2$	$2cLs(2x) = [cLs(x)]^2 +$ $+[sLs(x)]^2$
$Sh(2x) = 2\ sh(x)ch(x)$	$\frac{1}{\sqrt{5}} sFs(2x) = sFs(x)cFs(x)$	$sLs(2x) = sLs(x)cLs(x)$
$[ch(x)]^{(n)} = \begin{cases} sh(x), & for\ \ n = 2k+1 \\ ch(x), & for\ \ n = 2k \end{cases}$	$[cFs(x)]^{(n)} = \begin{cases} (\ln(\tau))^n sFs(x), & for\ \ n = 2k+1 \\ (\ln(\tau))^n cFs(x), & for\ \ n = 2k \end{cases}$	$[cLs(x)]^{(n)} = \begin{cases} (\ln(\tau))^n sLs(x), & for\ \ n = 2k+1 \\ (\ln(\tau))^n cLs(x), & for\ \ n = 2k \end{cases}$

For example, the most important identity for the classical hyperbolic functions,

$$[ch(x)]^2 - [sh(x)]^2 = 1 \quad (80)$$

is written for the hyperbolic Fibonacci and Lucas functions as :

$$[cFs(x)]^2 - [sFs(x)]^2 = \frac{4}{5}, \qquad (81)$$

$$[cLs(x)]^2 - [sLs(x)]^2 = 4. \qquad (82)$$

Thus, the above symmetric hyperbolic functions retain all of the properties of classical hyperbolic functions (Table 7) while exhibiting new (recursive) properties characteristic for Fibonacci and Lucas numbers (Table 6). Thus, unlike the classical hyperbolic functions, the new hyperbolic functions have discrete analogues in the form of Fibonacci and Lucas numbers. According to (76) these functions agree with Fibonacci and Lucas numbers when the continuous variable x takes integer values. Note that identities (76) - (82), as well as other identities from Tables 6 and 7 emphasize the fundamental character of the hyperbolic Fibonacci and Lucas functions.

We predict that hyperbolic Fibonacci and Lucas functions will have great importance for the future development of the traditional Fibonacci and Lucas number theory [2-4]. They generalize Fibonacci and Lucas numbers to the continuous domain since Fibonacci and Lucas numbers are embedded in them. According to Table 6 each discrete identity for Fibonacci and Lucas numbers has its continuous analogue in the form of a corresponding identity for the hyperbolic Fibonacci and Lucas functions, and conversely. Therefore, the theory of the hyperbolic Fibonacci and Lucas functions is more general than traditional Fibonacci and Lucas number theory. **It results in the following important consequence**: **due to the introduction of the hyperbolic Fibonacci and Lucas functions, discrete Fibonacci and Lucas number theory [2-4] becomes a part of the continuous theory of hyperbolic Fibonacci and Lucas functions [5-8]! The introduction of the hyperbolic Fibonacci and Lucas functions is a new stage in the development of Fibonacci and Lucas number theory.**

Bodnar's geometry

As is well known, Fibonacci and Lucas numbers are the basis of *Law of Plant Phyllotaxis* [28]. According to this law, the number of the left and right spirals on the surface of *phyllotaxis objects* such as the pine cone, pineapple, cactus, head of sunflower, etc. are adjacent Fibonacci numbers, i.e., numbers whose ratios are,

$$\frac{F_{n+1}}{F_n}: \quad \frac{2}{1},\frac{3}{2},\frac{5}{3},\frac{8}{5},\frac{13}{8},\frac{21}{13},\ldots \to \tau = \frac{1+\sqrt{5}}{2}. \qquad (83)$$

These ratios characterize the symmetry of phyllotaxis. Therefore, every phyllotaxis pattern is characterized by one of these ratios. This ratio is called the *order of symmetry*.

After observing phyllotaxis patterns on the surface of a plant, it is natural to wonder how this pattern is formed during the growth of the plant. This question is at the basis of the *phyllotaxis puzzle* which is one of the most intriguing problems of botany. The key to this puzzle lies in the tendency of bioforms during their growth to change their symmetry orders according to (69). For example, the florets of a sunflower at the different levels of the same stalk have different symmetry orders, that is, the older disks have symmetry orders further out in the sequence. It means that during growth there is a natural change f the symmetry order and this change of symmetry is carried out according to the law:

$$\frac{2}{1} \to \frac{3}{2} \to \frac{5}{3} \to \frac{8}{5} \to \frac{13}{8} \to \frac{21}{13} \to \ldots \qquad (84)$$

A change of symmetry orders of phyllotaxis objects according to (30) is called *dynamic symmetry* [28]. Many scientists feel that the phyllotaxis phenomenon has more general importance.

For example, it is Vernadsky 's opinion that dynamic symmetry may be applicable to general problems in biology.

So, the phenomenon of the dynamic symmetry finds a special role in the geometry of phyllotaxis. We can assume that certain geometrical laws are lurking within the numerical regularity of (84). This phenomenon may be the secret of the phyllotaxis growth mechanism, and its expression could have great importance for the solution of the phyllotaxis problem. The Ukrainian researcher Oleg Bodnar has recently shed light on this problem [28]. Bodnar formulated a geometric theory of phyllotaxis**.** This theory is based on the hypothesis, that the geometry of phyllotaxis is hyperbolic, and the change of symmetry orders of phyllotaxis objects during their growth is based on the hyperbolic turn, the basic transforming motion of hyperbolic geometry. However, the main feature of Bodnar's geometry lies in its use of "golden" hyperbolic functions, which agree with the symmetric hyperbolic Fibonacci and Lucas functions described in this paper up to constant factors.

Bodnar's geometry shows that in parallel with the hyperbolic space based on the classical hyperbolic functions (Lobachevsky's hyperbolic geometry, Minkovsky's geometry, etc.), there is the "golden" hyperbolic space based on the hyperbolic Fibonacci and Lucas functions [5-8]. **The "golden" hyperbolic space exists objectively and independently of our awareness. This "hyperbolic world" persistently shows itself in Living Nature. In particular, it appears in pinecones, sunflower heads, pineapples, cacti, and inflorescences of various flowers in the form of the Fibonacci and Lucas spirals on the surfaces of these biological objects (the phyllotaxis law). Note that the hyperbolic Fibonacci and Lucas functions [5-8], which underlie phyllotaxis phenomena, are not "inventiveness" of Fibonacci-mathematicians because they reflect objectively the major mathematical law underlying the Living Nature geometry.**

10. Fibonacci matrices

The idea of the Q-matrix (16) allows the following generalization [23]. For a given $p=$ 0, 1, 2, 3, … consider a special square (p+1)x(p+1)- matrix, called Q_p*-matrix*:

$$Q_p = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \qquad (85)$$

It contains a $p\times p$-*identity* matrix bordered by the last row, which consists of 0's and a leading 1, and the first column, which consists of 0's embraced by a pair of 1's. We list the Q_p-matrices for p = 0, 1, 2, 3, 4:

$$Q_0 = (1); \qquad Q_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = Q; \qquad Q_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix};$$

$$Q_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \qquad Q_4 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The main result of reference [23] is a proof of the following expression for the n-th power of the Q_p-matrix:

$$Q_p^n = \begin{pmatrix} F_p(n+1) & F_p(n) & \cdots & F_p(n-p+2) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-p) & \cdots & F_p(n-2p+2) & F_p(n-2p+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_p(n-1) & F_p(n-2) & \cdots & F_p(n-p) & F_p(n-p-1) \\ F_p(n) & F_p(n-1) & \cdots & F_p(n-p+1) & F_p(n-p) \end{pmatrix} \tag{86}$$

where p =0, 1, 2, 3, …, n = 0, ±1, ±2, ±3, …, and elements of the matrix are the Fibonacci p-numbers.

Note that the class of matrices (86) has an interesting recursive property. If we cross out the last, that is, (p+1)-th column and the next to the last, that is, p-th row in the Q_p-matrix, then the matrix is reduced to matrix Q_{p-1}. This means that the determinant of the Q_p-matrix differs from the determinant of the Q_{p-1}-matrix only by its sign, i.e.,

$$\text{Det } Q_{p-1} = - \text{Det } Q_p . \tag{87}$$

It is easy to calculate a determinant of the Q-matrix (16):

$$\text{Det } Q = \text{Det}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = -1 \tag{88}$$

Taking into consideration (87) and (88) we can write:

$$\text{Det } Q_p = (-1)^p. \tag{89}$$

It follows from general matrix theory that,

$$\text{Det } Q_p^n = (-1)^{pn}. \tag{90}$$

where p = 0, 1, 2, 3, … ; n = 0, ±1, ±2, ±3, … .

It is clear that matrices (85) and (86) can be used for the future progress of Fibonacci research. Here expression (90) can be considered to be a wide generalization of the "Cassini formula" (18). For example, for the case p=2 the generalized "Cassini formula" takes the form:

$$\text{Det } Q_2^n = F_2(n+1)[F_2(n-2)F_2(n-2) - F_2(n-1)F_2(n-3)] + F_2(n)[F_2(n)F_2(n-3) - F_2(n-1)F_2(n-2)] + \\ + F_2(n-1)[F_2(n-1)F_2(n-1) - F_2(n)F_2(n-2)] = 1. \tag{91}$$

While the classical Cassini formula (18) gives a connection between three adjacent Fibonacci numbers, formula (91) connects arbitrary five adjacent Fibonacci 2-numbers $F_2(n-3)$, $F_2(n-2)$, $F_2(n-1)$, $F_2(n)$ and $F_2(n+1)$ for any integer value of n.

It is clear that there is an infinite number of the generalized Cassini formulas similar to (91) for p=1, 2, 3, ...

11. The "golden" matrices

Stakhov has introduced a class of the "*golden" matrices* [14, 26]. Represent now matrix (17) in the form of two matrices, one for even (n=2k) and the other for odd (n=2k+1) values of n:

$$Q^{2k} = \begin{pmatrix} F_{2k+1} & F_{2k} \\ F_{2k} & F_{2k--1} \end{pmatrix} \tag{92}$$

$$Q^{2k+1} = \begin{pmatrix} F_{2k+2} & F_{2k+1} \\ F_{2k+1} & F_{2k} \end{pmatrix} \tag{93}$$

Using relation (76), we can write the matrices (92), (93) in terms of symmetric hyperbolic Fibonacci functions (72), (73):

$$Q^{2k} = \begin{pmatrix} cFs(2k+1) & sFs(2k) \\ sFs(2k) & cFs(2k-1) \end{pmatrix} \tag{94}$$

$$Q^{2k+1} = \begin{pmatrix} sFs(2k+2) & cFs(2k+1) \\ cFs(2k+1) & sFs(2k) \end{pmatrix} \tag{95}$$

where k is an integer.

Substituting discrete variable k in the matrices (94), (95) by the continuous variable x results in two unusual matrices that are functions of x:

$$Q^{2x} = \begin{pmatrix} cFs(2x+1) & sFs(2x) \\ sFs(2x) & cFs(2x-1) \end{pmatrix} \tag{96}$$

$$Q^{2x+1} = \begin{pmatrix} sFs(2x+2) & cFs(2x+1) \\ cFs(2x+1) & sFs(2x) \end{pmatrix} \tag{97}$$

It is clear that matrices (96) and (97) are generalizations of the Q-matrix (17) for the continuous domain. These matrices have a number of unusual mathematical properties. For example, for when $x = \frac{1}{4}$ the matrix (96) takes the following form:

$$Q^{\frac{1}{2}} = \sqrt{Q} = \begin{pmatrix} cFs(\frac{3}{2}) & sFs(\frac{1}{2}) \\ sFs(\frac{1}{2}) & cFs(-\frac{1}{2}) \end{pmatrix} \tag{98}$$

It is difficult to imagine what is meant by the "square root" of the Q-matrix», but just such a "Fibonacci fantasy" follows from (98)!

Using properties (78) and (79) of the symmetric hyperbolic functions we can compute the determinants of the matrices (96) and (97):

$$\text{Det } Q^{2x} = 1 \tag{99}$$

$$\text{Det } Q^{2x+1} = -1 \tag{100}$$

Therefore these determinants are independent on x and equal identically to 1 in absolute value. In fact, the identities (99) and (100) are generalizations of the Cassini formula to the continuous domain!

12. Algorithmic measurement theory and Fibonacci *p*-codes

The first optimization problem in measurement theory

As is well known measurement theory has a long history. Its origin is connected to the "*incommensurable segments"* discovery made by Pythagoreans at investigation of the ratio of the square diagonal to its side. This discovery caused the first crisis in mathematics foundations and resulted to appearance of *irrational numbers*.

In 1202 the first optimization problem appeared in measurement theory. The famous Italian mathematician Fibonacci was the author of this problem. This problem is called the *"problem of choosing the best system of standard weights"* or *Bashet-Mendeleev's problem* (in the Russian mathematical literature [29]) .

The essence of the problem consists in the following [29]. Let it be necessary to weigh any unknown weight Q in the range from 0 up to Q_{max} using n standard weights

$$\{q_1, q_2, ..., q_n\}, \tag{101}$$

where $q_1 = 1$ is a measurement unit; $q_i = k_i \times q_i$; k_i is any natural number.

It is clear that the maximum weight Q_{max} is equal to the sum of all standard weights, i.e.

$$Q_{max} = q_1 + q_2 + ... + q_n = (k_1 + k_2 + ... + k_n)\, q_1 \tag{102}$$

Then it appears the problem to find the *optimal system of the standard weights,* i.e. such standard weight system (101), which ensures the maximum value of Q_{max} given by (18) among all possible variants of (102). In this case we have to chose such variant of the standard weights (101) that it would be possible to compose any multiple by q_1 weight Q using the standard weights (101) taking each of them separately.

As is well known there are two variants of this problem [9]. For the former case we can place the standard weights only on the free cup of the balance, for the latter case we can place them on two cups of the balance.

The optimal solution for the former case is given by the "binary" system of standard weights, i.e.

$$\{1, 2, 4, 8, 16, ..., 2^{n-1}\} \tag{103}$$

Notice that the measurement algorithm based on the "binary" system "generates" so-called *"binary" measurement algorithm* that is used widely in the measurement practice. Notice that the "binary" algorithm "generates" the *"binary" number system* that underlies modern computers and information technology.

The "Asymmetry Principle of Measurement"

The analysis of the "binary" algorithm by using the balance model (Fig. 7) allows to discover one surprising measurement property having a general character for all thinkable measuring that are reduced to the comparison of the measurable weight Q with any standard weights.

Consider now a process of weighing the weight Q on the balance using the "binary" system of standard weights. At the first step of the "binary algorithm" the largest standard weight 2^{n-1} places on the free cup of the balance (Fig. 7-a). After the first step one may appear two situations for the cases $2^{n-1} < Q$ (Fig. 7-a) and $2^{n-1} \geq Q$ (Fig. 7-b). In the former case (Fig. 7-a) the second step is to place the next large standard weight 2^{n-2} to the free cup of the balance. In the latter case the "weigher" should perform two operations, i.e. remove the previous standard weight 2^{n-1} from the free cup of the balance (Fig. 7-b) so that the balance returns to the initial position (Fig. 7-c). Then the next standard weight 2^{n-2} places on the free cup of the balance (Fig. 7-c).

One can readily see that the both considered cases differ between themselves by their "complexity". In fact, in the former case the "weigher" performs only one operation, i.e. he places the next standard weight 2^{n-2} on the free cup of the balance. In the latter case the "weigher's" actions are determined by two reasons. First he has to remove the previous standard weight 2^{n-1} from the free cup of the balance and then to wait when the balance returns to the initial position. After the returning the balance to the initial position (Fig.7-c) the "weigher" places the next standard weight 2^{n-2} on the free cup of the balance (Fig. 7-c).

The discovered property of measurement was called the *"Asymmetry Principle of Measurement"* [9].

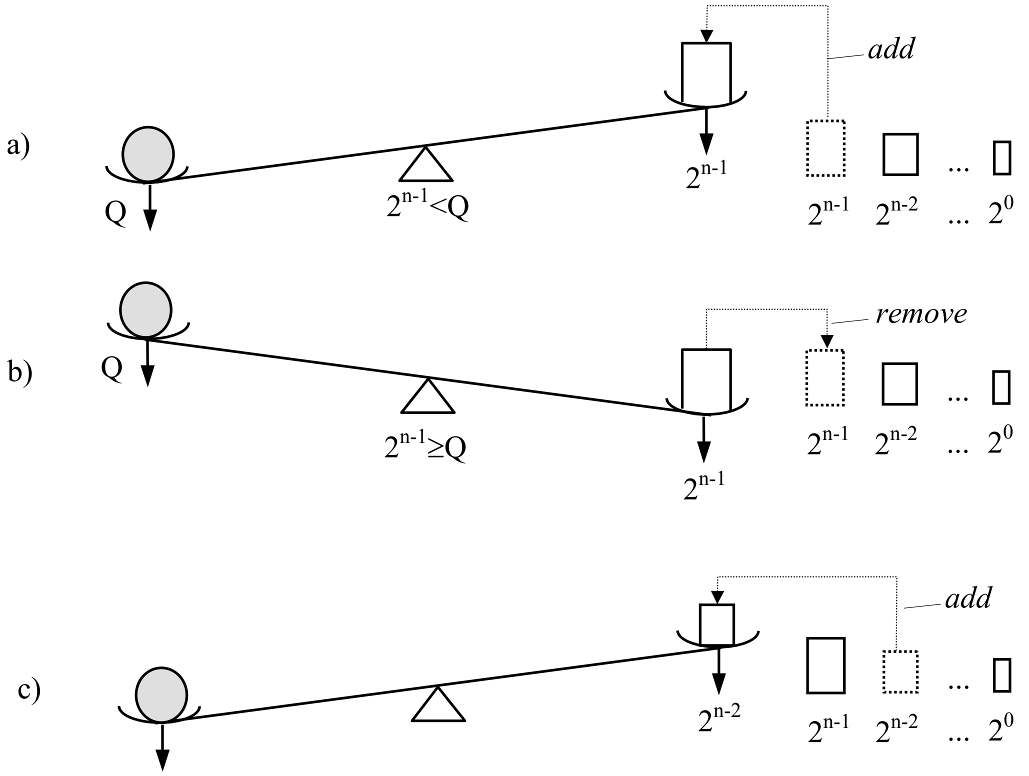

Figure 7. The "Asymmetry Principle of Measurement"

Fibonacci's measurement algorithm

Application of the "Asymmetry Principle of Measurement" to Bashet-Mendeleev's problem had led to the unexpected result. It is proved [9] that for a given p=0, 1, 2, 3, … (p is an "inertness" of the balance, that is, a time necessary for returning the balance to the initial state) the optimal solution of the problem is reduced to the following system of standard weights:

$$\{F_p(1), F_p(2), ..., F_p(i), …, F_p(n)\} \quad (104)$$

where $F_p(i)$ is the Fibonacci p-number given by (38), (39).

Fibonacci p-codes

The Fibonacci's measurement algorithms are isomorphic to the following positional representation of natural number N:

$$N = a_nF_p(n) + a_{n-1}F_p(n-1) + ... + a_iF_p(i) + ... + a_1F_p(1), \quad (105)$$

where $a_i\in\{0, 1\}$ is the binary numeral of the i^{th} digit of the code (105); n is the digit number of the code (105); $F_p(i)$ is the i^{th} digit weight calculated in accordance with the recurrent relation (38) at the seeds (39).

A positional representation of the natural number N in the form (105) is called *Fibonacci p-code* [9]. The abridged notation of the Fibonacci p-code (105) has the following form:

$$N = a_n\, a_{n-1} \dots a_i \dots a_1. \qquad (106)$$

Notice that the notion of the Fibonacci p-code includes an infinite number of the different methods of the binary representations of the kind (106) because every number p "generates" its own Fibonacci p-code (p = 0, 1, 2, 3, ...).

Consider now the partial cases of the Fibonacci p-code (105). For the case p = 0 the Fibonacci p-numbers are reduced to the binary numbers: 1, 2, 4, 8, 16, 32, …, 2^{i-1} … , that is,

$$F_0(i) = 2^{i-1} \qquad (107)$$

Substituting (107) into the formula (105) we get:

$$N = a_n 2^{n-1} + a_{n-1} 2^{n-2} . + ... + a_i 2^{i-1} . + ... + a_1 2^0 . \qquad (108)$$

This means that for the case p=0 the Fibonacci p-code (91) is reduced to the classical binary representation of natural numbers.

Let p=1. For this case the Fibonacci p-numbers $F_p(n)$ are reduced to the classical Fibonacci numbers F_n. It is clear that Fibonacci p-code for this case is reduced to Zeckendorf's representation [3]:

$$N = a_n F_n + a_{n-1} F_{n-1} + ... + a_i F_i + ... + a_1 F_1, \qquad (109)$$

Consider now the partial case $p = \infty$. In this case every Fibonacci p-number is equal to 1, i.e. for any arbitrary i we have:

$$F_p(i) = 1.$$

Then the sum (91) takes the following form:

$$N = \underbrace{1 + 1 + ... + 1}_{N} . \qquad (110)$$

Thus, the Fibonacci p-code given by (105) is a very wide generalization of the binary code (108) and Zeckendorf's representation (109) that are the partial cases of the Fibonacci p-code (105) for the cases p=0 and p=1 respectively. On the other hand, the Fibonacci p-code (105) includes in itself the so-called "unitary" code (110) as another extreme case for $p = \infty$.

13. Number systems with irrational radices

Bergman's number system

In 1957 the young American mathematician George Bergman published a paper *A number system with an irrational base* [30] in *Mathematics Magazine.* He suggested the following unusual positional number system:

$$A = \sum_i a_i \tau^i , \qquad (111)$$

where A is a real number, a_i is binary numeral (0 or 1) of the i-th digit, $i = 0, \pm1, \pm2, \pm3,$ τ^i is the weight of the i-th digit, τ is the base of number system (111).

At the first sight, there does not exist any distinction between the formula (111) and the formulas for the canonic positional number system, for example, the "binary" number system, but it is only at the first sight. A principal distinction of the number system (111) from the canonical positional number systems consists in the fact that the irrational number $\tau = \frac{1+\sqrt{5}}{2}$ (the golden ratio) is used as a radix of the number system (111). That is why Bergman called the sum (111) *number system with an irrational base* or *τ- system*. Although Bergman's paper [30] contained the result of a principal importance for the number system theory however in that period this paper simply did be noted neither by mathematicians nor by engineers. And in conclusion of his paper [30] George Bergman wrote: *"I do not know of any useful application for systems such as this, except as a mental exercise and pastime, though it may be of some service in algebraic number theory".*

Possibly the number system with irrational base developed by George Bergman in 1957 is the most important mathematical discovery in the field of number systems after the discovery of positional principle of number representation (Babylon, 2000 B.C.) and decimal number system (India, 5th century). This number system turns over our ideas about number systems, moreover, our ideas about the correlation between integer and irrational numbers. Before Bergman's system all traditional positional number systems had natural bases (10 for the decimal number system, 60 for the Babylonian sexagecimal number system, 2 for the "binary" number system and so on). In the number system (111) for the first time the irrational number $\tau = \frac{1+\sqrt{5}}{2}$ is used as a base of positional number system. This means that the irrational number $\tau = \frac{1+\sqrt{5}}{2}$ is the beginning of all numbers, including natural numbers and all real numbers! It is very surprising that George Bergman made his unusual mathematical discovery in the age of 12 years!

A generalization of Bergman's number system

Bergman's number system (111) allows the following generalization. Consider an infinite sequence of the golden *p*-proportions powers:

$$\{\ldots, \tau_p^n, \tau_p^{n-1}, \ldots, \tau_p^0 = 1, \tau_p^{-1}, \ldots, \tau_p^{-k}, \ldots\} \qquad (112)$$

where τ_p is the golden *p*-proportion, a real root of the golden algebraic equation (24). Remind that all the golden *p*-proportion powers in (112) are connected between themselves by the remarkable identity (25).

Stakhov introduced in [12] the following positional method of number representation based on the set (112):

$$A = \sum_i a_i \tau_p^i \ , \qquad (113)$$

where a_i is the binary numeral of the i^{th} digit; τ_p^i is the weight of the i^{th} digit; τ_p is the radix of the number system (113), $i = 0, \pm 1, \pm 2, \pm 3, \ldots$.

Notice that the formula (113) gives a theoretically infinite number of the binary positional representations of real numbers because every $p = 0, 1, 2, 3, \ldots$ "generates" its own method of the binary positional number representation in the form (113).

A radix is one of the fundamental notions of positional number system. As the analysis of the sum (113) shows, the radix of the number system (113) is the golden *p*-proportion τ_p . That is why the representation of real number *A* in the form (113) was called in [12] *code of the golden p-proportion of real number A.*

Notice that except of the case $p = 0$ ($\tau_0 = 2$) all the rest golden *p*-proportions τ_p are irrational numbers. It follows from this fact that the code of the golden *p*-proportion given by the sum (113) are the binary number systems with irrational radices for the case $p > 0$.

Consider now the partial cases of the golden p-proportion codes given by (113). For the case $p = 0$ we have: $\tau_p = \tau_0 = 2$ and therefore the golden p-proportion code (113) is reduced to the classical "binary" number system

$$A = \sum_i a_i 2^i , \tag{114}$$

which underlies modern computers.

For the case $p = 1$ the golden p-proportion τ_p is reduced to the classical golden proportion (5) introduced by Euclid and then the golden p-proportion code (113) is reduced to Bergman's number system (111).

The abridged notation of the sum (113) has the following form:

$$A = a_n\, a_{n-1} \ldots a_1\, a_0 \,,\, a_{-1}\, a_{-2} \ldots a_{-k}. \tag{115}$$

Here the comma separates the abridged notation (115) into two parts. The left part corresponds to the binary digits with nonnegative indices; the right part corresponds to the binary digits with negative indices.

A representation of the powers of the golden p-proportions

Consider now some peculiarities of the number representation in the number system (113). The numbers being the powers of the golden p-proportions are represented in the number system (113) very easy. In particular, the radix of the number system (99) is represented in traditional form, i.e.

$$\tau_p = 10. \tag{116}$$

The number $1 = \tau_p^0$ has the following code representation:

$$1 = \tau_p^0 = 1{,}0 \tag{117}$$

The positive and negative powers of the golden p-proportion are represented in the following forms:

$$\begin{array}{ll} \tau_p^1 = 10 & \tau_p^{-1} = 0.1 \\ \tau_p^2 = 100 & \tau_p^{-2} = 0.01 \\ \tau_p^3 = 1000 & \tau_p^{-3} = 0.001. \end{array} \tag{118}$$

Let us consider now a representation of arbitrary real number A in Bergman's number system (111) as follows:

$$A = \tau^4 + \tau^3 + \tau^0 + \tau^{-1} + \tau^{-2} + \tau^{-5} \tag{119}$$

It is clear that the sum (119) has the following abridged "binary" representation:

$$A = 1\,1\,0\,0\,1,\,1\,1\,0\,0\,1. \tag{120}$$

Using Binet formula (13) we can write the sum (120) as follows:

$$A = \frac{L_4 + F_4\sqrt{5}}{2} + \frac{L_3 + F_3\sqrt{5}}{2} + \frac{L_0 + F_0\sqrt{5}}{2} + \frac{L_{-1} + F_{-1}\sqrt{5}}{2} + \frac{L_{-2} + F_{-2}\sqrt{5}}{2} + \frac{L_{-5} + F_{-5}\sqrt{5}}{2} \tag{121}$$

Substituting the values of the Fibonacci and Lucas numbers taken from Table 1:

$$L_4 = 7;\ L_3 = 4;\ L_0 = 2;\ L_{-1} = -1;\ L_{-2} = 3;\ L_{-5} = -11;$$

$F_4 = 3$; $F_3 = 2$; $F_0 = 0$; $F_{-1} = 1$; $F_{-2} = -1$; $F_{-5} = 5$

into the sum (121) we will get the number A in the explicit form:

$$A = \frac{4+10\sqrt{5}}{2} = 2 + 5\sqrt{5}. \tag{122}$$

Notice that all real numbers given by (118), (119) are irrational numbers! But according to (118), (119) they are represented as a finite set of the binary numerals. This means that Bergman's number system (111) and its generalization (113) allow representing some irrational numbers (in particular, the powers of the golden p-proportions and their sums) by the finite set of binary numerals that is absolutely impossible in the classical positional number systems! This statement is the first unusual property of the number systems (111), (113) and their fundamental distinction from the traditional positional number systems with integer radices (binary, decimal, etc.).

14. New properties of natural numbers

The "golden" representations of natural numbers

Consider representations of natural numbers in Bergman's number system (111) and the number system (113), that is,

$$N = \sum_i a_i \tau^i \tag{123}$$

$$N = \sum_i a_i \tau_p^i \tag{124}$$

Representations of natural number N in the form (123) and (124) is called *τ- and τ_p-codes of natural number N* respectively.

The following unusual theorems for τ- and τ_p-codes are proved in [15].

Theorem 1. All natural numbers can be represented in the τ-code (123) as a finite sum of the golden proportion powers.

Theorem 2. For a given $p > 0$ all natural numbers can be represented in the τ_p-code (124) as a finite sum of the golden p-proportion powers.

Z- and D-properties of natural numbers, F- and L-codes

The sums (123) and (124) can become a source of new number-theoretical results. These properties for the τ-code (123) are proved in [15].

Theorem 3 (*Z*-property of natural number). If we represent any natural number N in the τ-code (123) and then substitute every power of the golden proportion τ^i in the expression (123) by the Fibonacci number F_i, where the index i takes its values from the set $\{0, \pm1, \pm2, \pm3, \ldots\}$, then the sum arising as result of such substitution is equal to 0 identically independently on the initial natural number N, that is,

$$\sum_i a_i F_i = 0. \tag{125}$$

Theorem 4 (*D*-property). If we represent any natural number N in the τ-code (123) and then substitute every power of the golden proportion τ^i in the expression (123) by the Lucas number L_i, where the index i takes its values from the set $\{0, \pm1, \pm2, \pm3, \ldots\}$, then the sum arising as result of such substitution is equal to $2N$ identically independently on the initial natural number N, that is,

$$\sum_i a_i L_i = 2N. \tag{126}$$

Theorem 5 (*F*-code of natural number). If we represent any natural number N in the τ-code (123) and then substitute every power of the golden proportion τ^i in the expression (123) by the Fibonacci number F_{i+1}, where the index i takes its values from the set {0, ±1, ±2, ±3, …}, then the sum arising as result of such substitution is equal to the initial natural number N:

$$N = \sum_i a_i F_{i+1} \tag{127}$$

The sum (127) is called *F-code of natural number N.*

Theorem 6 (*L*-code of natural number). If we represent any natural number N in the τ-code (123) and then substitute every power of the golden proportion τ^i in the expression (123) by the Lucas number L_{i+1}, where the index i takes its values from the set {0, ±1, ±2, ±3, …}, then the sum arising as result of such substitution is equal to the initial natural number N:

$$N = \sum_i a_i L_{i+1} \tag{128}$$

The sum (128) is called *L-code of natural number N.*

Thus, there are three different methods of the positional representation of one and the same natural numbers, *τ-code* (123), *F-code* (127) and *L-code* (128). Notice that the binary numerals a_i in the sums (123), (127), (128) coincide. This means that abridged notations of the τ-code, F-code and L-code of one and the same natural number N are identical.

As example we consider the representation of the decimal number 10 in Bergman's number system:

$$10 = 1\ 0\ 1\ 0\ 0,\ 0\ 1\ 0\ 1. \tag{129}$$

Notice that the code representation (129) is an abridged notation in the τ-code (129) of the following sum:

$$10 = \tau^4 + \tau^2 + \tau^{-2} + \tau^{-4}. \tag{130}$$

Using Binet formula (13) we can represent the sum (130) as follows:

$$10 = \tau^4 + \tau^2 + \tau^{-2} + \tau^{-4} = \frac{L_4 + F_4\sqrt{5}}{2} + \frac{L_2 + F_2\sqrt{5}}{2} + \frac{L_{-2} + F_{-2}\sqrt{5}}{2} + \frac{L_{-4} + F_{-4}\sqrt{5}}{2}. \tag{131}$$

If we take into consideration the following correlations connecting the Fibonacci and Lucas numbers

$$L_{-2} = L_2;\quad L_{-4} = L_4;\quad F_{-2} = -F_2;\quad F_{-4} = -F_4$$

we will get from (131) the following expression:

$$10 = \frac{2(L_4 + L_2)}{2} = L_4 + L_2 = 7 + 3.$$

Consider now the interpretation of the code representation (129) as the F- and L-codes:

$$10 = F_5 + F_3 + F_{-1} + F_{-3} = 5 + 2 + 1 + 2$$

$$10 = L_5 + L_3 + L_{-1} + L_{-3} = 11 + 4 - 1 - 4.$$

15. Ternary mirror-symmetric arithmetic

Stakhov in [22] developed an original computer arithmetic called *ternary mirror-symmetric arithmetic*. This one is a synthesis of the τ-code (123) and ternary symmetrical number system used by the Russian engineer Nikolay Brousentsov in "Setun" computer that is the first in computer history ternary computer based on the "Brousentsov's Ternary Principle" [22].

It is proved in [22] that every natural number can be represented in the form:

$$N = \sum_i c_i \tau^{2i} \quad , \tag{132}$$

where c_i is the ternary numeral { $\bar{1}$, 0, 1} of the i^{th} digit; τ^{2i} is the weight of the i^{th} digit; τ^2 is the radix of the number system (132).

Notice that we can use (132) for the positional ternary representation of all integers, positive and negative. For example the ternary representation (132) of the number 5 has the following form:

$$\begin{array}{ccccccc} & & 2 & 1 & 0 & -1 & -2 \\ 5 & = & 1 & \bar{1} & 1, & \bar{1} & 1 \end{array} \tag{133}$$

Considering the ternary representation of the number 5 we can find that the left-hand part ($1\ \bar{1}$) of the ternary representation (133) is mirror-symmetric to its right-hand part ($\bar{1}\ 1$) relative to the 0-th digit. This property of the *mirror symmetry* has a general character and is valid for all integers. Taking into consideration the *mirror-symmetric property*, the ternary representation (132) was called *ternary mirror-symmetric representation* [22].

It follows from (132) that the radix of the number system (132) is the square of the golden proportion:

$$\tau^2 = \frac{3+\sqrt{5}}{2} \approx 2,618.$$

This means that the number system (132) is a number system with an irrational radix.

The radix of the number system (132) has the following traditional representation:

$$\tau^2 = 10.$$

The following identities for the golden proportion powers underlie the ternary mirror-symmetric arithmetic:

$$2\tau^{2k} = \tau^{2(k+1)} - \tau^{2k} + \tau^{2(k-1)}; \tag{134}$$

$$3\tau^{2k} = \tau^{2(k+1)} + 0 + \tau^{2(k-1)}; \tag{135}$$

$$4\tau^{2k} = \tau^{2(k+1)} + \tau^{2k} + \tau^{2(k-1)}, \tag{136}$$

where $k = 0, \pm1, \pm2, \pm3, \ldots$.

The identity (134) is a mathematical basis for the mirror-symmetric addition of two single-digit ternary digits and gives the rule of the carry formation (Table 8).

Table 8. Mirror-symmetric addition

b_k \ a_k	$\bar{1}$	0	1
$\bar{1}$	$\bar{1}\,1\,\bar{1}$	$\bar{1}$	0
0	$\bar{1}$	0	1
1	0	1	$1\,\bar{1}\,1$

As follows from Table 8, the main peculiarity of the ternary mirror-symmetric addition consists in the fact that at the addition of the significant ternary digits of the same sign (1+1 and $\bar{1} + \bar{1}$) the carries spread symmetrically to the higher and the lower digits of the ternary mirror-symmetric representation.

The subtraction of the two mirror-symmetric numbers N_1 - N_2 is reduced to the mirror-symmetric addition if we represent their difference in the following form:

$$N_1 - N_2 = N_1 + (-N_2). \tag{137}$$

It follows from (137) that before subtraction it is necessary to take a "ternary inversion" of the subtrahend N_2 according to the rule:

$$1 \to \bar{1};\ 0 \to 0;\ \bar{1} \to 1.$$

The following trivial identity for the golden proportion powers underlies the mirror-symmetric multiplication:

$$\tau^{2n} \times \tau^{2m} = \tau^{2(n+m)}.$$

A rule of the mirror-symmetric multiplication of two single-digit ternary mirror-symmetric numbers is given in Table 9.

Table 9. Mirror-symmetric multiplication

b_k \ a_k	$\bar{1}$	0	1
$\bar{1}$	1	0	$\bar{1}$
0	0	0	0
1	$\bar{1}$	1	1

The ternary mirror-symmetric arithmetic [22] has a number of unique mathematical properties: (1) positive and negative integers are represented in direct form; (2) all ternary mirror-symmetrical representations of integers have "mirror-symmetrical form"; (3) all arithmetical operations are performed in the "direct" form; (4) the result of every arithmetical operation is represented in the "mirror-symmetrical form". It is clear that the ternary mirror-symmetric arithmetic can lead to the original computer projects based on the "Ternary Principle" (ternary representations of numbers, ternary logic, ternary memory element, flip-flap-flop) [22].

16. A new coding theory based on the Fibonacci matrices

Represent an initial message in the form of the square matrix M of the size $(p+1)\times(p+1)$, where p=0, 1, 2, 3, …. Choice now the Fibonacci Q_p-matrix Q_p^n of the size $(p+1)\times(p+1)$ as a *coding matrix* and its inverse matrix Q_p^{-n} of the same size as a *decoding matrix.*

Consider now the following transformations based on matrix multiplication (see Table 10).

Table 10. Fibonacci coding/decoding method

Coding	Decoding
$M\times Q_p^n = E$	$E\times Q_p^{-n} = M$

We will name a transformation $M\times Q_p^n = E$ as *Fibonacci coding* and a transformation $E\times Q_p^{-n} = M$ as *Fibonacci decoding.* We will name the matrix E as *code matrix.*

The coding/decoding method given by Table 10 provides an infinite variants of possible transformation of the initial matrix M into the code matrix E because every Fibonacci coding matrix Q_p^n and its inverse matrix Q_p^{-n} (p=1, 2, 3, …; n=1, 2, 3, …) "generate" their own Fibonacci coding/decoding method according to Table 10. Notice that for the case p=0 the matrix Q_p reduces to the trivial matrix Q_0 = (1) and for this case the coding/decoding method given by Table 10 is trivial. For the case p=1 the matrix Q_p is reduced to the classical Q-matrix (17).

Calculate now a determinant of the code matrix E:

$$\text{Det } E = \text{Det } [M\times Q_p^n] = \text{Det } M\times \text{Det } Q_p^n \tag{138}$$

Taking into consideration (90) we can write the expression (138) in the form:

$$\text{Det } E = \text{Det } M\times(-1)^{pn}. \tag{139}$$

where $p = 0, 1, 2, 3, \ldots$; $n = 0, \pm1, \pm2, \pm3, \ldots$.

We can use the identity (139) as the main "checking correlation" of the Fibonacci coding/decoding method given by Table 10. For this purpose we will calculate Det M and then send it to the "communication channel" after the code matrix E. Using Det M we can detect and correct errors, which can appear in the code matrix E under influence of noise in the "communication channel".

As is shown in [24, 25] the Fibonacci coding/decoding method differs from the classical algebraic codes by the following peculiarities:

(1) The Fibonacci coding/decoding method is reduced to matrix multiplication, i.e. to well-known algebraic operation, which can be carried out well in modern computers.
(2) The main practical peculiarity of the method is the fact that large information units, in particular matrix elements, are objects of error detection and correction. It is proved that the simplest Fibonacci coding method (p=1) can guarantee the restoration of all "erroneous" code (2×2)-matrices having "single", "double" and "triple" errors.
(3) The correct ability of the method for the simplest case p=1 is equal 93,33% that exceeds essentially all well-known correcting codes.

17. A new kind of cryptography

Consider now the following encryption/decryption algorithms based on matrix multiplication (see Table 11).

Table 11. Encryption/decryption algorithm

Encryption	Decription
$M\times Q^{2x} = E_1(x)$	$E_1(x)\times Q^{-2x} = M$
$M\times Q^{2x+1} = E_2(x)$	$E_2(x)\times Q^{-2x+1} = M$

Here M is the *plaintext* represented in the form of the square 2×2-matrix M:

$$M = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \tag{140}$$

$E_1(x)$, $E_2(x)$ are *ciphertexts,* Q^{2x}, Q^{2x+1} are the *enciphering matrices* (96), (97); Q^{-2x} and Q^{-2x+1} are the *deciphering matrices* having the following forms [26]:

$$Q^{-2x} = \begin{pmatrix} cFs(2x-1) & -sFs(2x) \\ -sFs(2x) & cFs(2x+1) \end{pmatrix} \tag{141}$$

$$Q^{-2x+1} = \begin{pmatrix} -sFs(2x) & cFs(2x+1) \\ cFs(2x+1) & -sFs(2x+2) \end{pmatrix} \tag{142}$$

We can use a variable x as a *cryptographic key.* This means that in dependence on the value of the key x there is an infinite number of transformation of the plaintext M into the ciphertext $E(x)$.

If we calculate determinants of the matrices $E_1(x)$, $E_2(x)$ and take into consideration the identities (99), (100) we will get the following identities connecting the matrix M with the matrices $E_1(x)$, $E_2(x)$:

$$\text{Det } E_1(x) = \text{Det } M \quad (143)$$

$$\text{Det } E_2(x) = - \text{Det } M \quad (144)$$

We can use the identities (143), (144) for checking a process of encryption and decryption what is important from practical point of view.

If we will use the existing asymmetrical cryptosystems for the transmission of the cryptographic key we can solve elegantly many cryptographic problems, namely:

(1) We can use the "golden" cryptographic system given by Table 11 for the fast transformation of the plaintext (140) into the ciphertext E and conversely according to Table 7.
(2) We can use the unique mathematical property of the "golden" cryptography given by (143), (144) for checking the encryption and decryption algorithms.

This means that using the "golden" cryptography method (Table 11) we can design *fast, simple for technical realization and reliable cryptosystems,* which can be used for cryptographic protection of communication systems working in real time.

18. Conclusion

In conclusion we return back again to Euclid's *Elements.* There are no doubts that Euclid's *Elements* is one of the most outstanding mathematical works that give rise to many modern mathematical concepts and theories. It is enough to remind *Euclidean geometry,* a concept of *prime numbers*, *Euclidean algorithm* (Theorem VII, 2), *Eudox' theory of irrationality*, which gave the beginning of many fascinating branches of modern mathematics, in particular, geometry and number theory. A study of Euclid's V postulate led to the discovery of the *non-Euclidean geometry* (Nikolay Lobachevsky and others). However, we should note that many original Euclidean concepts and ideas did not get due development in modern mathematics until now. It is necessary to recognize, that in this respect a destiny of the problem of *division in the extreme and ratio* (DEMR), formulated by Euclid in Theorem II, 11, appeared most difficult. As is shown by the Canadian mathematician Roger Herz-Fishler in his remarkable book [1], a problem of the DEMR literally "pierces" Euclid's *Elements* starting from Book I and finishing with Book XIII. And although after Euclid many outstanding mathematicians, including Luca Paccioli, Johannes Kepler, Binet, Lucas, Vorobyov, Verner Hoggat, paid their attention to the DEMR (the golden section), however the golden section remains for many mathematicians the "beautiful fairy tale", not having any relation to serious mathematics.

The following words about the DEMR belonged to Johannes Kepler, who was not only a Great astronomer but also a Great mathematician: *"Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious stone".* Theorem of Pythagoras is known for all pupils; however a majority of the so-called "educated" peoples has a vague representation about the division of a line into extreme and mean ratio. This means that the golden section, one of the great treasures of geometry, is ignored by the contemporary mathematical education.

In 1884 the Great mathematician Felix Klein published a book *Lectures on Icosahedron and Solution of the 5th degree Equations* [31]. Klein treats a regular icosahedron as a geometric object, from which the branches of the five mathematical theories follow, namely, *geometry, Galois' theory, group theory, invariants theory and differential equations.* Klein's main idea is extremely simple: *"Each unique geometrical object is somehow or other connected to properties of the regular icosahedron"*. Unfortunately, in contrast to his *Erlangen program* Klein's idea about the icosahedron as the main geometric object of mathematics did not get due recognition in mathematics. However, we should recognize that Klein genially predicted a role of the regular icosahedron in modern science. It was Platonic icosahedron and Archimedean truncated icosahedron, which became the main geometric idea of *quasi-crystals* (Dan Shechtman, 1982) [32] and *fullerenes* (Robert F. Curl, Harold W. Kroto and Richard E. Smalley, 1985, Nobel Prize of 1996) [33]. These and others outstanding scientific discoveries based on the golden section (DEMR) and regular polyhedra described in Euclud's *Elements* are a cause of a great interest in the DEMR in modern science.

As is shown in the present article, the golden section (DEMR) is a source of many interesting ideas and concepts of modern mathematics and computer science. The author developed his research in the following hierarchy:

Division in extreme and mean ratio (the golden section)
Fibonacci and Lucas numbers, Binet formulas, Q-matrix
↓
The golden p-sections and Fibonacci p-numbers
Generalized principle of the golden Section
The "golden" algebraic equations
Generalized Binet formulas for Fibonacci and Lucas p-numbers
Hyperbolic Fibonacci and Lucas functions
Fibonacci matrices based on the Fibonacci p-numbers
"Golden" matrices based on the hyperbolic Fibonacci functions
Algorithmic measurement theory and Fibonacci p-codes
Number systems with irrational radices
Z- and D-properties, τ-, F- and L-codes of natural numbers
Ternary mirror-symmetric arithmetic
A new coding theory based on the Fibonacci matrices
A new cryptography method based on the "golden" matrices

It seems to us that the dramatic history of the DEMR (the golden section), which continued over several millennia, may be concluded with a great triumph for the golden section in the beginning of the 21st century. Many outstanding scientific discoveries based on the golden section (quasi-crystals [32], fullerenes [33], "golden" genomatrices [34] and so on) give a reason to suppose that **the golden section may be a kind of some "metaphysical knowledge", "pre-number", or "universal code of Nature", which could become the foundation for the future development of science, in particular, mathematics, theoretical physics, genetics, and computer science.**

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