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Fundamentals of a new kind of mathematics based on the Golden Section

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Dedicated to Mohamed Salah El Nashie's 60th birthday

Abstract

The attempt of build up the Fundamentals of a new mathematical direction, which is called *Harmony Mathematics*, is addressed in the present article. The article has a "strategic" importance for development of computer science and theoretical physics.

> I want to know how God created this world. I am not interested in this or that phenomenon, in the spectrum of this or that element. I want to know His thoughts; the rest are details.
> Albert Einstein

1. Introduction

As is well known, mathematics is one of the outstanding creations of the human intellect; a result of centuries of intensive and continuous creative work of man's geniuses. What is the goal of mathematics? The answer is not simple. Probably, the goal of mathematics is to discover "mathematical laws of the Universe" and to construct models of the physical world. It is clear that the progress of the human society depends on the knowledge of these laws.

During historical progress, mankind realized that it is surrounded by a huge number of different "worlds": the "world" of geometric space, the "world" of mechanical and astronomical phenomena, the "world" of stochastic processes, the "world" of information, the "world" of electromagnetism, the "world" of botanical and biological phenomena, and the "world" of art, etc. For simulation and mathematical description of each of these "worlds", mathematicians created the appropriate mathematical theory most suitable to the phenomena and processes of this or that "world". To describe a geometric space, Euclid wrote his book *The Elements*. To simulate the mechanical and astronomical phenomena, Newton created the *theory of gravitation and differential and integral calculus*. *Maxwell's*

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theory was created to describe the electromagnetic phenomena; the theory of probabilities was created for simulation of the stochastic "world". In the 19th century, Lobachevsky created non-Euclidean geometry which is a deeper model of the geometric space. We can go on mentioning infinitely many such examples.

Modern mathematics is a complex set of different mathematical concepts and theories. One of the major problems of mathematical research is to find connections between separate mathematical theories. This always leads to the deepening of our knowledge about Nature and shows a deep connection between Nature and Universal laws.

Modern mathematics experienced a complex stage in its development. The prolonged crisis of its bases was connected to paradoxes in *Cantor's theory of infinite sets*. The passion of mathematicians for abstractions and generalizations broke the contact with natural sciences that are a source of mathematical origin. This has compelled many outstanding mathematicians of the 20th century to talk about a serious crisis in modern mathematics and even about its isolation from the general course of scientific and technical progress. In this connection the publication of the book *Mathematics, The Loss of Certainty* [1], written by Moris Kline, Professor Emeritus of Mathematics of Courant Institute of Mathematical Sciences (New York University), is symptomatic.

In this situation, the representatives of other scientific disciplines, namely, physics, chemistry, biology, engineering and even arts, began to develop what may be called *natural mathematics*, which can be used effectively for mathematical simulations of physical, biological, chemical, engineering and other processes. The idea of *soft mathematics* gained more and more attractiveness. Humanitarization of mathematics is being discussed as a tendency in the development of modern science [2]. In this connection the book *Meta-language of the Living Nature* [3] written by the famous Russian architect, Shevelev, can be considered as an attempt to create one more variant of *natural mathematics*.

Harmony Mathematics that was developed by the author for many years [4–22] belongs to a category of similar mathematical directions. In its sources this new mathematical theory goes back to the Pythagorean Doctrine about Numerical Harmony of the Universe. Since the antique period, many outstanding scientists and thinkers like Leonardo da Vinci, Luca Pacioli, Johannes Kepler, Leibnitz, Zeizing, Binet, Lucas, Einstein, Vernadsky, Losev, Florensky paid a great attention to this scientific doctrine.

The main goal of the present article is to state the fundamentals of the Harmony Mathematics [4–22], that is, to describe its basic concepts and theories and to discuss its applications in modern science.

2. A combinatorial approach to the Harmony Mathematics

For the first time the consecutive representation about the world as internally contradictory and harmoniously whole was developed by the ancient Greeks. In Ancient Greece the studying of the Beauty essence was formed into a independent scientific branch called *aesthetics*, which was inseparable from cosmology in ancient sciences. The Pythagorean doctrine about the Universe Harmony had a particular importance for the aesthetics history because it was the first attempt to understand the concept of Harmony. Pythagoreans put forward the idea about the harmonious construction of the Universe including not only human nature, but also all Cosmos. According to Pythagoreans, Harmony is internal connection of things and phenomena, without which the Cosmos can not exist. At last, according to Pythagoreans, Harmony has numerical expression; it is integrally connected to the concept of number.

Despite many thousand-years experience of studying the Harmony problem it today belongs to the category of the most difficult scientific problems [23]. The main reason is that Harmony is a very complex scientific concept, which expresses not only quantitative, but also the qualitative aspects of the studied phenomenon.

As is know, mathematics studies the quantitative aspect of this or that phenomenon. And starting with the mathematical analysis of the Harmony concept, we should concentrate our attention on the quantitative aspects of Harmony. What is the quantitative aspect of this concept? To answer this question we will start with the analysis of the origin and meaning of the word "Harmony". As is known the word "Harmony" has a Greek origin. The Greek word αρμουια means *connection*, *consent*.

There are various definitions of the Harmony concept. However, the majority of them lead to the following definition given in "The Great Soviet Encyclopedia":

> Harmony is proportionality of parts and the whole, combination of the various components of the object in the uniform organic whole. The internal ordering and measure obtain in Harmony external expression.

The analysis of the word "Harmony" and its definition shows, that the most important, key notions, which underlie this concept, are the following: *connection*, *consent*, *combination*, *ordering*.

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Table 1
Pascal triangle

									1									
								1		1								
							1		2		1							
						1		3		3		1						
					1		4		6		4		1					
				1		5		10		10		5		1				
			1		6		15		20		15		6		1			
		1		7		21		35		35		21		7		1		
	1		8		28		56		70		56		28		8		1	
1		9		36		84		126		126		84		36		9		1

There is a question: which branch of mathematics studies these concepts? The search of the answer to this question leads us to the combinatorial analysis. As is known, ''the combinatorial analysis studies the various kinds of combinations and connections, which can be formed from the elements of some finite set. The term 'combinatorial' occurs from the Latin word combinary—to combine, connect'' [24].

It follows from this consideration that both the Latin word *combinary* and the Greek word $\alpha\rho\mu o\nu\iota\alpha$ is essence have the some meaning, namely, *combination* and *connection*. It gives us the basis to put forward the assumption that exactly the ''Laws of Combinatorial analysis'' can be used for analysis of the Harmony concept from the quantitative point of view.

As is known [24], the combinatorial analysis, as a mathematical discipline, originated in the 17th century. The famous mathematician and physicist, Blaise Pascal (1623–1662), in his work *Treatise on the arithmetic triangle* (1665) set forth the doctrine of the binomial coefficients. One of the most well-known mathematical formulas that is named as Newton's binomial formula is connected to binomial coefficients:

$$(a+b)^n = a^n + C_n^1 a^{n-1} b + C_n^2 a^{n-2} b^2 + \cdots + C_n^k a^{n-k} b^k + \cdots + C_n^{n-1} a b^{n-1} + b^n. \tag{1}$$

The numerical coefficients C_n^k in the formula (1) are called *binomial coefficients*.

Note that the name of Newton's binomial formula is not true from a historical point of view because this formula was known long before Newton, by many scientists in different countries including Tartalia, Fermat, and Pascal. Newton's merit is that he proved that this formula is valid for the case of any real number n, rational or irrational, positive or negative.

It was proved that binomial coefficients have a number of remarkable mathematical properties namely

$$C_n^0 = C_n^n = 1, \tag{2}$$

$$C_n^k = C_n^{n-k}, \tag{3}$$

$$C_{n+1}^k = C_n^{k-1} + C_n^k. \tag{4}$$

The identity (4) is called *Pascal's law*. Using identity (4), Pascal discovered a graceful way of calculation of the binomial coefficients based on their representation in the form of a special numerical table named as *Pascal triangle* (Table 1).

Thus, the main concepts that underlay the Harmony notion and the combinatorial analysis in essence coincide. And this give us a right to use the combinatorial analysis as the initial mathematical theory for the analysis of the Harmony concept. This is the main idea of the present research, in which the author made an attempt to create the *Fundamentals of the Harmony Mathematics* that is based on the binomial coefficients and Pascal triangle. And though the combinatorial relations for Fibonacci numbers have been derived by the author already in his first book [4], the understanding of what these combinatorial relations mean is the main idea of the Harmony Mathematics that occured to the author more recently. This unexpected comprehension, that exactly the combinatorial analysis is a mathematical basis of the harmonious correlations in Nature, Science and Arts became an incentive motive of write this article.

3. A historical view on the development of the harmony concept: from Khesi-Ra and Pythagoras to El Nashie and Vladimirov

3.1. Khesi-Ra

In the beginning of the 20th century in Saqqara (Egypt), archeologists opened the crypt, in which the remnants of the Egyptian architect by name *Khesi-Ra* were buried. In the literature, this name often matches as Khesira. It is supposed that Khesi-Ra was the contemporary of Imhotep who lived in the period of the pharaoh Zoser (27th century BC) because in the crypt the printings of this pharaoh are found. The wood panels, which were covered with a magnificent carving, were extracted from the crypt along with different material things. In total there were in the crypt 11 panels; among them only five panels were preserved; the remaining panels were completely broken down because moisture got through reached to the crypt. On all the preserved panels, the architect Khesi-Ra, was surrounded with different things that have a symbolical significance. Since a long time the assigning of panels of the Khesi-Ra crypt was vague. At first the Egyptologists considered these panels as false doors. However, since 1960s the situation of the panels has been clearer. In the beginning of the 60s the Russian architect Shevelev, paid his attention to one of the panels and the batons, which the architect held in his hands, related between themselves as $1 : \sqrt{5}$, that is, as the small side and the diagonal of the rectangle with a side ratio of 1:2 (“two-adjacent squares”). This observation became a “launching pad” for the researches of the other Russian architect Shmelev, who arrived at the etiled geometrical analysis of Khesi-Ra’s panels. As a result he came to a sensational discovery described in the brochure *Phenomenon of Ancient Egypt* [25]:

> Now, after the comprehensive and argued analysis by the method of proportions we get good reason to assert that Khesi-Ra’s panels are the harmony rules encoded by geometry language... So, in our hands we have the concrete material evidences, which shows us by “plain text” the highest level of abstract thinking of the Ancient Egypt intellectuals. The artist, who carved the panels with amazing and jeweler accuracy and masterly ingenuity, demonstrated the rule of the Golden Section in its broadest range of variations. In outcome it was born the GOLDEN SYMPHONY presented by the ensemble of the highly artistic works, which testifies not only ingenious talents of their creator, but also verifies convincingly that the author was initiated into the secret of Harmony. This genius was of the Golden Business Craftsman by the name of Khesi-Ra.

And the last quotation from Shmelev’s brochure [25]: “It is necessary only to recognize, that the Ancient Egypt civilization is the super-civilization that was studied by us extremely superficially and this fact requires from us a qualitatively new approach to the studying of the richest heritage of the Ancient Egypt...”

3.2. Pythagoras

Pythagoras was born about 596 BC in Samos, Ionia, and died about 475 BC. He is possibly the most celebrated person in the history of science. His name is known to every person who has studied geometry and is aware of the Pythagorean Theorem. He was described as a famous philosopher and scientist, religious and ethical reformer, influential politician, was a “Demi-God” in the eyes of his followers and a charlatan in the eyes of some of his contemporaries; such characteristics are attributed to Pythagoras in the ancient literature. The coins bearing his image, made during 430–420 BC, testify to the exclusive popularity of Pythagoras during his lifetime. This tribute was unprecedented in the fifth century BC. Pythagoras was also the first among the Greek scientists for whom a special book was exclusively dedicated.

The Pythagorean doctrine concerned harmony, geometry, number theory, astronomy, etc. The Pythagoreans most appreciated results are those obtained in the theory of harmony, because they confirmed the idea that “the numbers determine everything”. Pythagoras was an enthusiast famous Golden Section. Some ancient scientists assume that Pythagoras borrowed the concept of the Golden Section from the Babylonians. Proclus, the ancient Greek mathematician, attributes the discovery of the five regular polyhedrons to Pythagoras.

Why was Pythagoras so popular during his lifetime? The answer to this question is provided by some interesting facts from his biography [26]. In the article dedicated to Pythagoras [26] it is noted that, “according to the legend Pythagoras went away to Egypt and lived there 22 years to study the knowledge of the eastern scientists. After studying of all sciences of the Egyptians, including mathematics, he moved to Babylon, where he lived 12 years and studied the scientific knowledge of the Babylonian priests. The legend attributes to Pythagoras a visit to India. It is very probable as Ionia and India then had business relations. On returning home (about 530 BC) Pythagoras attempted to organize his philosophical school. However for unknown reasons he soon abandoned Samos and settled in Croton (a Greek colony in the north of Italy). Here Pythagoras organized the school, which acted almost thirty years”.

Thus, the outstanding role of Pythagoras in the development of Greek science was his fulfillment *of an historical mission that would ultimately transfer the knowledge of the Egyptian and Babylonian priests to the culture of Ancient Greece*. Thanks to Pythagoras, who was without any doubt, one of the most learned thinkers of his time, Greek science gained a tremendous volume of knowledge in the fields of philosophy, mathematics and natural sciences. The addition of such knowledge, substantially contributed to the rapid progress of the Ancient Greek culture.

3.3. Plato

The Great Greek philosopher *Plato*, was born in Athens, about 427 BC, and died there about 347 BC. Plato's works, perhaps the most popular and influential philosophic writings ever published, consist of a series of dialogues in which the discussions between Socrates and others are presented with infinite charm. Plato did believe that mathematics in its ideal form could still be applied to the heavens. The heavenly bodies, he believed, exhibited perfect geometric form. This idea is expressed by Plato most clearly in a dialogue called *Timaeus* in which he presents his scheme of the Universe. He describes the five (and only five) possible regular solids—that is, those with equal faces and with all equal lines and angles, that are formed by those faces. These are the 4-sided tetrahedron, the 6-sided hexahedron (or cube), the 8-sided octahedron, the 12-sided dodecahedron, and the 20-sided icosahedrons. Four of the five regular solids, according to Plato, represented the four basic elements: Fire, Earth, Air, and Water; while the dodecahedron represented the Universe as a whole. These solids were first discovered by the Pythagoreans, but the notoriety of the *Timaeus* dialogue ensured that they would always be referred to as the *Platonic solids*. It is important to note that both the dodecahedron and the icosahedrons are connected to the Golden Section.

3.4. Euclid

Euclid of Alexandria *was born about 325 BC and died about 265 BC in Alexandria, Egypt. He is the most prominent mathematician of antiquity. Euclid's most famous word is his treatise on mathematics known as, **The Elements***. This book was a compilation of ancient mathematical knowledge that became the centre of mathematical teaching for 2000 years. Euclid may not have been a first class mathematician, but his famous, *The Elements*, brought to him the glory of being the leading mathematics teacher of antiquity or perhaps of all time. ***The Elements** is divided into 13 books. Books one to six deal with plane geometry. Books seven to nine deal with number theory. Books ten deals with the theory of irrational numbers. Books 11 to 13 deal with three-dimensional geometry. The Elements ends with book 13, which discusses the properties of the five regular polyhedra and gives proof that there are precisely five. The Elements* gives us for the first time a geometric problem of a *division of line in the extreme and middle ratio*. In modern mathematics this problem is known as the *Golden Section problem*. The concluding book of, *The Elements*, book 13, was devoted to the Platonic Solids theory that became a source of the widespread hypothesis, that a description of the theory of Platonic Solids theory that became a source of the widespread hypothesis, that a description of the theory of Platonic Solids, express the Universe Harmony, as the main goal of the *The Elements*.

3.5. Fibonacci

The "Middle Ages" in our consciousness, is associated with the concept of inquisition orgy, campfires, on which witches and heretics are incinerated, and crusades for God's body. The science in those times obviously was not "in a center of society attention". In this connection the appearance of the mathematical book "Liber abaci" written in 1202 by the Italian mathematician *Leonardo Pisano* (by the nickname of *Fibonacci*) was an important event in the "scientific life of society". Fibonacci was born in 1170 and died in 1250 in Italy. He was educated in North Africa where his father held a diplomatic post. Fibonacci studied mathematical in Bugia and traveled widely with his father. He recognized enormous advantages of the mathematical systems used in the countries they visited. The main book *Liber abaci*, published in 1202 after Fibonacci's return to Italy, was based on the arithmetic and algebric knowledge that Fibonacci had accumulated during his travels. The book introduced the Hindu-Arabic decimal system and use of Arabic numerals into Europe. Certainly many of the problems that Fibonacci considers in *Lider abaci* were similar to those appearing in Arab sources. Though Fibonacci had one of the brightest mathematical minds in the history of West-European mathematics, his contribution to mathematics is belittled undeservedly. A significance of Fibonacci's mathematical creativity for mathematics is assessed properly by the Russian mathematician, Prof. Vasiljev, in his book *Integer Number* (1919):

> The works of the learned Pisa merchant were so above the level of mathematical knowledge even of the scientists of that time, that their influence on the mathematical literature becomes noticeable only two centuries after his death at the end of the 15th century, when many of his theorems and problems are entered in the works of Luca Pacioli, who was a professor at many Italian universities and the Leonardo da Vinci friend, and in the beginning of the 16th century, when the group of the talented Italian mathematicians: Ferro, Cardano, Tartalia, Ferrari by the solution of the cubical and biquadrate equations gave the beginning of higher algebra.

Fibonacci's role in the development of West-European mathematics can be compared with Pythagoras' role in the development of Greek science. Like Pythagoras, the historical role of Fibonacci in West-European mathematics is that his mathematical books promoted transferring or Arabian mathematical knowledge to West-European science that created fundamentals for further development of the West-European mathematics.

Ironically, Fibonacci, who made an outstanding contribution to the development of mathematics, became known in modern mathematics only as the author of the interesting numerical sequence called *Fibonacci numbers*:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots \tag{5}$$

This numerical sequence was obtained by Fibonacci as the solution for the famous problem of *rabbits reproduction*. The formulation and solution of this problem is considered as Fibonacci's main contribution to the development of combinatorial analysis. It is specifically with the help of this problem Fibonacci anticipated the method of *recurrence relations* that can be considered as one of the most powerful methods of combinatorial analysis. Fibonacci's recurrence relation,

$$F(n) = F(n-1) + F(n-2), \tag{6}$$

obtained by Fibonacci as the solution for this problem, is considered to be the first recurrence relation in mathematics history.

Note that for the initial terms

$$F(0) = 0, \qquad F(1) = 1, \tag{7}$$

the recurrence relation (6) generates Fibonacci numbers (5).

3.6. Pacioli

The outstanding Italian mathematician and learned monk *Luca Pacioli* was born in 1445 in the provincial town of Borgo de San-Sepolcoro (which translated from Italian means a not very joyous "City of the Sacred Coffin"). He died in 1517. Pacioli was known as professor of many Italian universities. Pacioli's pedagogical work was combined with scientific activity. He wrote two encyclopedic mathematical books. In 1494 he published the book entitled *The Sum of arithmetic's geometry*, *doctrine about proportions and relations*. All of the books' material is divided into two parts; the former part is dedicated to arithmetic and algebra, the latter one to geometry. One part of the book is dedicated to the problems of mathematics' application to commercial business and in this part his book was a continuation of Fibonacci's famous book "Liber abaci" (1202). In essence, this mathematical work was a sum of the mathematical knowledge of the Renaissance. Pacioli was a friend and scientific advisor of Leonardo da Vinci. Under Leonardo da Vinci's direct influence, Pacioli wrote his second famous book *De Divina Proportione*. This book, published by Pacioli in 1509, rendered a noticeable influence on his contemporaries. Pacioli's folio was one of the first examples of the Italian book-printing art. The historical significance of the book is that it was the first mathematical book dedicated to the Golden Section. The book was illustrated with 60 magnificent figures drawn by Leonardo da Vinci. The book consists of there parts: in the first part the properties of the Golden Section are given, the second part is dedicated to regular polyhedra, the third one is dedicated to the applications of the Golden Section in architecture.

3.7. Kepler

The outstanding astronomer and mathematician *Johannes Kepler* was born is 1571 and died in 1630. In 1591, he enrolled in the Tubingen Academy where he got quite a good mathematical education. Here the future great astronomer was acquainted with the heliocentric system of Nicola Copernicus. After graduation from the Academy, Kepler got a Master's degree and was then appointed as mathematics teacher in the Graz High School (Austria). His first book with the intriguing title, *Misterium Cosmographium*, was published by Kepler in 1596 at the age of 25. Here he developed a very original geometric model of the Solar system that was based on the Platonic Solids. Though his model appeared erroneous, Kepler remained confident in this scientific result an always considered this model as one of his highest scientific achievements. Kepler was a consecutive follower of the Golden Section, Platonic Solids and the Pythagorean Doctrine about the numerical Harmony of the Universe. Kepler expressed his admiration of the Golden Section in the following words:

> Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious stone.

Kepler's works terminated the epoch of "scientific romanticism", the epoch of Harmony and the Golden Section, that was inherent to the Renaissance. But on the other hand, his scientific works signaled the beginning of a new science, which started to develop from the works of Descartes, Galileo and Newton. With Kepler's death in 1630 the Golden Section, which he considered one of the "geometry treasures", was forgotten. This strange oblivion continued for almost two centuries. Interest in the Golden Section was not revived again until the 19th century.

3.8. Pascal

The famous French physicist and mathematician *Blaise Pascal* was born in 1623 and died in 1662. Pascal was a versatile researcher. He invented the first digital calculator (1642–1645) called Pascaline that resembled the mechanical calculator of the 1940s. He developed the Conic Sections and proved important theorems is projective geometry. In, *The Generation of Conic Sections* (1648), Pascal considered *Conics* generated by central projection of a circle. From May 1653 Pascal worked on physics and wrote *Treatise on the Equilibrium of Liquids* (1653), in which he explained *Pascal's law of pressure*.

As we mentioned above, Pascal was one of the founders of combinatorial analysis. In his works *Treatise on the arithmetic triangle* (1665) he stated a doctrine on the binomial coefficients and for the first time constructed *Pascal triangle*.

3.9. Riccati

The Italian 17th century mathematician *Vincenzo Riccati* was born in 1707 and died in 1775. He was the second son of the Italian mathematician, Jacopo Riccati. Vincenzo continued his father's work on integration and differential equations. Vincenzo studied hyperbolic functions. He found their relation to the exponential function and their derivatives. Also he introduced the standard designation for hyperbolic functions in the following form:

$$\mathrm{sh}(x) = \frac{e^x - e^{-x}}{2}, \qquad \mathrm{ch}(x) = \frac{e^x + e^{-x}}{2}. \tag{8}$$

A great role of the hyperbolic functions (8) developed by Vincenzo Riccati was understood later when the Russian geometer Nikolay Lobachevsky discovered non-Euclidean geometry and the German mathematician Herman Minkovsky gave a geometric interpretation of Einstein's special theory of relativity.

3.10. Lobachevsky

A history of mathematics shows that there exists a "strange" tradition in mathematics regarding the outstanding mathematical discoveries. Many mathematicians (even the very famous), as a rule, are not able to properly access the mathematical achievements of their contemporaries. The revolutionary mathematical discoveries either remain unnoticed or are subjected to ridicule by their contemporaries and only 20–30 years later these discoveries begin to be recognized and admired.

The name of the French mathematician *Evarist Galois* is well-known in mathematics. His mathematical works were the origin for modern algebra. His basic mathematical works, which were named later *Galois Theory* in his honor, were developed by him between the ages of 16–18. Galois sent his works to the Paris Academy of Sciences. However, even the greatest French mathematicians Cauchy and Fourier could not understand Galois' works. According to the legend, academician Cauchy threw out Galois' mathematical manuscripts in the garbage. Evariste Galois was killed in 1832 during duel at the age of 21 years. Twenty-four years after Galois' death, the famous French mathematician Joseph Liouville edited some of Galois' manuscripts and published them with a glowing commentary. And ever since the Evariste Galois had been ranked as a mathematical genius.

A history of the non-Euclidean geometry, the outstanding discovery of the ingenious Russian geometer *Nikolay Lobachevsky*, however, is considered to be one of the most shameful pages in the history of the Russian academic science. Nikolay Lobachevsky was born in 1792 and died in 1856. His major mathematical research is connected to the fifth postulate of Euclidean geometry. On February 11 of 1826, in the session of the Department of Physics-Mathematical Sciences at Kazan University, Lobachevsky requested that his word on the fundamentals of a new geometry be sent to the referees at the St-Petersburg Academy of Sciences. The famous Russian mathematician academician Ostrogradsky gave a sharply negative review to Lobachevsky's article. In an anonymous article published in the journal "The Son of Fatherland" Lobachevsky's article was named as the geometric "speculations" of the "Kazan's rector, Mr. Lobachevsky". During his entire lifetime, Lobachevsky was subjected to ridicule by members of the official Russian academic community of that period. Lobachevsky's recognition eventually came from the West only after the genius German mathematician Gauss properly assessed Lobachevsky's works in geometry. It was Gauss who proposed that Lobachevsky be chosen as a Corresponding Member of the Gettingen scientific society.

The creation of the non-Euclidean geometry was Lobachevsky's main achievement. Lobachevsky used the hyperbolic functions (8) to describe the geometric relations for his geometrical studies. Therefore, Lobachevsky's geometry is called *hyperbolic geometry*.

3.11. Lucas

The famous 19th century French mathematician *Francois Edouard Anatole Lucas* was born in 1842 and died in 1891. The major works of Lucas are in number theory. In 1878, Lucas gave the criterion for the definition of the primality of Mersenn's numbers of the kind $2^n - 1$. Using his method, Lucas established that the number of $2^{127} - 1$ is a prime one. For over 75 years this number was the greatest prime number known in science. Also he found the 12th "perfect number" and formulated a number of interesting mathematical problems. Lucas was the first scientist who understood the role of the binary number system in the progress of science. He believed that with the help of machines, it is more convenient to perform the summation of numbers binary number system than in decimal ones. He contributed substantially to the development of the Fibonacci numbers theory. He introduced into mathematics the name of *Fibonacci numbers*. After Lucas' works, the mathematical articles in this field started to reproduce as "Fibonacci's rabbits". Lucas is the author of the following recurrent numerical sequence:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, \ldots, \tag{9}$$

called *Lucas numbers* in his honor. Note that Lucas numbers $L(n)$ is given by the recurrence relation

$$L(n) = L(n-1) + L(n-2), \tag{10}$$

with the initial terms

$$L(0) = 2, \qquad L(1) = 1. \tag{11}$$

3.12. Binet

The other famous 19th century French mathematician, *Jacques Philippe Marie Binet*, was born in 1776 and died in 1856. He contributed significantly to the study of mathematics. Binet developed the matrix theory. In 1812, he found a rule of matrix multiplication and this discovery glorified his name more than any other works of his. Also he introduced the notion of the *Beta function* and developed the linear differential equations theory. However, the Binet formulas that connect the Golden Ratio to Fibonacci and Lucas numbers are his most important contribution to the Fibonacci number theory [27–29]:

$$F(n) = \frac{\tau^n - \tau^{-n}(-1)^n}{\sqrt{5}}, \tag{12}$$

$$L(n) = \tau^n + \tau^{-n}(-1)^n, \tag{13}$$

where $\tau = \frac{1+\sqrt{5}}{2} \approx 1618$ is the *Golden Proportion or the Golden Ratio*, $L(n)$, $L(n)$ are Fibonacci and Lucas numbers given for the positive and negative values of the discrete variable n ($n = 0, \pm1, \pm2, \pm3, \ldots$). If we take into consideration that Binet formulas (12) and (13) connect integers (Finbonacci and Lucas numbers) to irrationals (the Golden Ratio) we can conclude that the formulas (12) and (13) have great importance in number theory.

3.13. Klein

Felix Klein was born on 25/4/1849 and delighted in pointing out that each of the day (5^2), month (2^2), and year (43^2) were the square of a prime. He died in 1925. Klein's main works were dedicated to the non-Euclidean geometry, theory, continuous groups, theory of algebraic equations, theory of elliptic and automorphic functions. His ideas in the field of geometry were stated by Klein in the work *Comparative consideration of new geometrical researches* (1872) known under the title *Erlangen Program*. According to Klein, each geometry is the invariant theory for the special group transformation that allows to pass from one type of geometry to other. Euclidean geometry is the science about the metric group invariants, a projective geometry about the projective group invariants, etc. The classification of transformation groups gives us the classification of the geometries. The proof of the non-Euclidean geometry consistency is considered as the essential Klein's achievement.

Klein's researches concern also regular polyhedrons. His book *The Lectures about a regular icosahedrons and solution of the 5th degree equations*, published in 1884, is dedicated to this problem. Though the book is dedicated to the solution of the fifth degree algebraic equations, the main idea of the book is much deeper. Its main purpose is to show a role of the *Platonic Solids*, in particular *Regular Icosahedron*, in the development of mathematical science. According to Klein, the issue of mathematics runs up widely and freely by sheets of the different theories. But there are mathematical objects, in which some sheets converge. Their geometry binds the sheets and allows to understand a general mathematical sense of the miscellaneous theories. The regular icosahedron, in Klein's opinion, is such a mathematical object. *Klein treats the regular icosahedron as the mathematical object, from which the branches of the five mathematical theories follow, namely geometry, Galois' theory, group theory, invariants theory and differential equations.*

What is the significance of Klein's ideas from the point of view of the Harmony concept? First of all, we see that the regular icosahedron, one of the Platonic Solids, is selected as the geometric object integrating the main sheets of mathematics. But the regular dodecahedron is based on the Golden Section! It follows from here that exactly the Golden Section is the main geometrical idea, which, according to Klein, can bind all branches of mathematics. Klein's contemporaries could not understand and assess properly a revolutionary sense of Klein's "Icosahedron" idea. Its significance was assessed properly 100 years later, namely in 1984, when the Israeli scientist Dan Shechtman published that articles verifying an existence of special alloys (called quasi-crystals) that have so-called "Icosahedron" symmetry, that is, the fifth order symmetry, which is strictly forbidden by classic crystallography.

3.14. Einstein

The Great physicist *Albert Einstein* was born in 1879 and died in 1955. Einstein made many scientific discoveries in physics and cosmology. But his *special theory of relativity* suggested in 1905 is considered as his highest scientific achievement. After 1905, Einstein continued to work in this area. He made important contributions to quantum theory, but he sought to extend the special theory of relativity to phenomenas involving acceleration. Einstein received the Nobel Prize in 1921 but not for relativity rather for his 1905 work on the photoelectric effect. It was Einstein who believed in the Universe Harmony. He said: "Religiousness of a scientist consists in an enthusiastic worship for laws of Harmony".

3.15. Minkovsky

The famous German mathematician *Herman Minkovsky* was born in 1864 and died in 1909. In 1908, three years after the promulgation of Einstein's special relativity theory, Herman Minkovsky presented the geometric substantiation of Einstein's special theory of relativity. Minkovsky's idea is characterized by two essential peculiarities. First of all, his geometric spatial–temporary model is four-dimensional: in it the spatial and temporary coordinates are connected in a common coordinate system. A position of the material point in Minkovsdy's space is determined by the point $M(x,y,z,t)$ called *World Point*. Secondly, the geometric connection between the spatial and temporary coordinates in Minkovsky's system has a non-Euclidean character; that is, the given model reflects certain peculiar properties of real space–time, which cannot be described in the frameworks of the "traditional" Euclidean geometry. Geometrically a connection between spatial (x) and temporary (t) coordinated in Minkovsky's space is given with the help of the *hyperbolic functions* (8). He gave an original geometric interpretation of the well-known *Lorenz formulas*. According to Minkovsky, Lorenz's transformation can be described in terms of hyperbolic geometry. In essence, Minkovsky's geometry uncovers a hyperbolic nature of all the mathematical formulas Einstein's relativity theory. At the same time if follows from here that all the analytical relations of Einstein's relativity theory objectively reflect the non-Euclidean character of the physical space–time.

3.16. Zeckendorf

Many number theorists know about *Zeckendorf's sum*:

$$N = a_nF(n) + a_{n-1}F(n-1) + \cdots + a_iF(i) + \cdots + a_1F(1), \tag{14}$$

where a_i is a binary numeral 0 and 1; $F(i)$ is Fibonacci number given by (6) and (7), but few know about the person after whom this sum is named. *Edouard Zeckendorf* (1901–1983) was a Colonel of the Belgian Medical Corps and an amateur of mathematics. He was a Dutch citizen. Edouard was born in Liege, and in 1925 he qualified as a medical doctor at the University of Liege and then became a Belgian Army officer. Before 1930, he also obtained a license for dental surgery. In 1940, Zeckendorf's was taken prisoner by the Germans, and in that capacity, he provided medical care until 1945. Zeckendorf's sum (14) originates with an article published in 1939. Each positive integer has a unique representation as a sum of two non-adjacent Fibonacci numbers. Zeckendorf's representation (14) can be compared with binary representation. Numerous articles published in, The Fibonacci Quarterly discuss Zeckendorf's sum and give its many generalizations.

3.17. Bergman

The American mathematician *George Bergman* was born in 1945. He works now as Professor in the Mathematics Department, University of California. In 1957, he made an original discovery in number system theory called *Bergman's number system*:

$$A = \sum_i a_i \tau^i, \tag{15}$$

where a_i is a binary numeral 0 or 1; τ^i is the weight of the ith digit of the number system (15); $\tau = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio; $i = 0, \pm1, \pm2, \pm3, \ldots$ It is surprising that George Bergman made his mathematical discovery in the age of 12! Despite his young age, Bergman's article [30] was published in the very authoritative mathematical journal "Mathematics Magazine", and the well-known public Journal "Times" had even interviewed the mathematical genius of America.

3.18. Hoggat

The American mathematician *Verner Email Hoggat* was born in 1921 and died in 1981. Verner Hoggatt, together with Brother *Alfred Brousseau*, published the first volume of *The Fibonacci Quarterly* in 1963, thereby founding the Fibonacci Association. The *Quarterly* has grown into a well-recognized number theory journal. On April 4, 1969, the *TIME* Magazine reported about the phenomenal growth of the Fibonacci Association. That same year, Houghton Mifflin published Hoggat's book, *Fibonacci and Lucas Numbers* [28], perhaps the world's best introduction to the Fibonacci numbers theory. Verner Hoggat was one of the World's first mathematicians who felt the approaching of "Fibonacci's era". He was also one the first mathematicians who paid attention to the *Q-matrix*,

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{16}$$

By raising to its nth power it shows its connections to Fibonacci numbers:

$$Q^n = \begin{pmatrix} F(n+1) & F(n) \\ F(n) & F(n-1) \end{pmatrix}, \tag{17}$$

where $n = 0, \pm1, \pm2, \pm3, \ldots$

3.19. Penrose

The British physicist and mathematician Sir *Roger Penrose* was born in 1931. Sir Roger Penrose, a professor of mathematics at the University of Oxford in England, pursues an active interest in geometrical puzzles. He was fascinated with a field of geometry known as *tessellation*, the covering of a surface with tiles of prescribed shapes. He wanted to cover a flat surface with tiles so that there were no gaps and no overlaps. He found non-periodic tilings, called *Penrose tilings*, with only two tiles. The Penrose tiles consist of two rhombi with angles 72° and 36°. The edges of the rhombi are all of equal length. They originated from *pentagram*. If we follow a few strict rules about how to place these rhombi together, you will find non-periodic patterns that have *pentagonal symmetry*.

3.20. Shechtman

The famous Israeli physicist *Dan Shechtman* is the author of revolutionary discovery in crystallography [31]. The Icosahedral Phase, discovered by Prof. Shechtman in 1982 is the focus of his research. The Icosahedral phase, as the first structure in the field of quasi-periodic crystals, was discovered in aluminum metal alloys. Its crystallography is unique, since it has the icosahedral group symmetry, and its atomic order is quasi-periodic in contrast to periodic order found in previously known crystals. The intriguing questions regarding structure and properties of this new class of materials draw a broad interdisciplinary scientific interest. The field has become an active major communication arena for physicists, material's scientists, mathematicians, crystallographers and others.

3.21. El Nashie

The outstanding Egyptian physicist *Mohamed-Saladdin El Naschie* was born in 1943 in Cairo. He is Professor of Frankfurt Institute of Advanced Studies, Member of the founding Board of Trustees, University of Frankfurt, Germany, Professor Astrophysics, Faculty of Science, University of Cairo, Egypt. He was a follower, student and friend of the Great scientist and Nobel Laureate Ilya Prigogin. El Nashie's areas of scientific interests are: stability, bifurcation, atomic-engineering, nonlinear dynamics, chaos, fractals, high-energy particle physics, quantum mechanics. Recently he made a new discovery in the area of high-energy particle physics. So far, we have already known experimentally 60 particles, but Mohamed El Nashie's theory (*E*–infinity theory) shows the most probable number of elementary particles in the standard model is 69 particles. In his theory, Prof. El Naschie used the theory of fractals and the number theory (where he gives a big role to the Golden Ratio that can be represented as infinite fraction) to postulate that the Universe has infinite of dimensions. According to [32], El Nashie's theory will lead to Nobel Prize if experimentally verified. Prof. El Nashie is a follower of the Golden Ratio and shows in his works [31–39] that the Golden Ratio plays outstanding role in physical researches.

3.22. Vladimirov

The famous Russian physicist *Yuri Vladimirov* was born in 1938. Currently he is Professor of the Theoretical Physics Department of Moscow University. His areas of scientific interests are: theoretical physics; theory of gravitation and quantization of gravitation; multivariate geometrical models of physical interactions, the theory of direct inter-partial interaction, the incorporated theory of space–time and physical interactions; philosophical problems of fundamental theoretical physics. In article [40] he suggested an original model of *quark icosahedron*. In 2002 he published the book "Metaphysics" [41]. The concluding paragraph of the book is named *Vineberg's angle and the Golden section*. He ends the book by the following words: "Thus, it is possible to assert that in the theory of electroweak interactions there are relations that are approximately coincident with the 'Golden Section' the play an important role in the various areas of science and art".

4. The main mathematical concepts and theories of the Harmony Mathematics

*4.1. Generalized Fibonacci numbers or Fibonacci **p**-numbers*

Let us consider a *rectangular Pascal triangle* (see Table 2) and its "modified" variants given by Tables 3 and 4.

Note that the binomial coefficient C_n^k in Table 2 is on the intersection of the *n*th column ($n = 0, 1, 2, 3, \ldots$) and the *k*th row ($k = 0, 1, 2, 3, \ldots$) of the Pascal triangle. If we sum the binomial coefficients of Table 2 by columns, we will get the "binary" sequence: 1, 2, 3, 4, 8, 16, ..., 2^n, ... In the combinatorial analysis this result is expressed in the form of the following elegant identity:

$$2^n = C_n^0 + C_n^1 + C_n^2 + \cdots + C_n^n. \tag{18}$$

Let us shift now every row of the initial Pascal triangle (Table 2) by *p* columns ($p = 1, 2, 3, \ldots$) to the right, relative to the preceding row and consider the "modified" Pascal triangles called *Pascal p-triangles*. Table 3 gives *Pascal 1-triangle* and Table 4 gives *Pascal 2-triangle*.

If we sum the binomial coefficients of the Pascal 1-triangle (Table 3) we will arrive unexpectedly at the Fibonacci numbers *F*(*n*) that are given with the recurrence relation (6) and (7). It is easy to deduce the following identity that connects Fibonacci numbers *F*(*n*) and binomial coefficients:

Table 2
Rectangular Pascal triangle

1	1	1	1	1	1	1	1	1	1
	1	2	3	4	5	6	7	8	9
		1	3	6	10	15	21	28	36
			1	4	10	20	35	56	84
				1	5	15	35	70	126
					1	6	21	56	126
						1	7	28	84
							1	8	36
								1	9
									1
1	2	4	8	16	32	64	128	256	512

Table 3
Pascal 1-triangle

1	1	1	1	1	1	1	1	1	1	1	1
		1	2	3	4	5	6	7	8	9	10
				1	3	6	10	15	21	28	36
						1	4	10	20	35	56
								1	5	15	35
										1	6
1	1	2	3	5	8	13	21	34	55	89	144

Table 4
Pascal 2-triangle

1	1	1	1	1	1	1	1	1	1	1
			1	2	3	4	5	6	7	8
						1	3	6	10	15
									1	4
1	1	1	2	3	4	6	9	13	18	28

$$F(n+1) = C_n^0 + C_{n-1}^1 + C_{n-2}^2 + \cdots + C_{m+r}^m, \tag{19}$$

where n, m and r are connected by the following relation:

$$n = 2m + r. \tag{20}$$

The result (19) is well-known in Fibonacci numbers theory [27–29]. However this result has a principal importance for development of the Harmony mathematics because it confirms fruitfulness of the combinatorial approach to the Harmony Mathematics. This result is a brilliant demonstration of the deep connection between Fibonacci numbers that express Harmony of Nature and the Art and Pascal triangle.

If we sum the binomial coefficients of Pascal 2-triangle (Table 4) by columns, we will get a new recurrent numerical sequence 1, 1, 1, 2, 3, 4, 6, 9, 13, 28, 41, 60, ... that is given by the following recurrence relation:

$$F_2(n) = F_2(n-1) + F_2(n-3), \tag{21}$$

with the initial terms

$$F_2(1) = 0, \qquad F_2(1) = F_2(2) = 1. \tag{22}$$

We will name this recurrence numerical sequence *Fibonacci 2-numbers*.

It is easy to deduce the following identity that connects Fibonacci 2-numbers $F_2(n)$ and binomial coefficients:

$$F_2(n+1) = C_n^0 + C_{n-2}^1 + C_{n-4}^2 + \cdots + C_{m+r}^m, \tag{23}$$

where n, m and r are connected by the following relation:

$$n = 3m + r. \tag{24}$$

If we sum by columns the binomial coefficients of the Pascal p-triangle, we will arrive at the numerical sequence that is expressed by the following recurrence relation [4]:

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \quad \text{where } n > p = 1, \tag{25}$$

with the initial terms

$$F_p(0) = 0, \qquad F_p(1) = F_p(2) = \cdots = F_p(p) = 1. \tag{26}$$

Note that the recurrence relation (25) with the initial terms (26) gives an infinite number of new numerical sequences. Moreover, the "binary" sequence 1, 2, 4, 8, 16, ..., is the special case of this sequence for $p = 0$ and the classical Fibonacci numbers (5) are the special case of this sequence for $p = 1$! We will name the numerical sequence generated by (25) and (26) as *Fibonacci p-numbers*.

It is easy to deduce the following identity that connects Fibonacci p-numbers $F_p(n)$ and binomial coefficients:

$$F_p(n+1) = C_n^0 + C_{n-p}^1 + C_{n-2p}^2 + \cdots + C_{m+r}^m, \tag{27}$$

where n, m and r are connected by the following relation:

$$n = (p+1)m + r. \tag{28}$$

4.2. The generalized golden ratios

If we consider the ratios of the adjacent Fibonacci p-numbers

$$F_p(2)/F_p(1), F_p(3)/F_p(2), \ldots, F_p(n)/F_p(n-1), \tag{29}$$

and then aim the sequence (29) for infinity, we will come to the "golden" algebraic equation [22]

$$x^{p+1} - x^p - 1 = 0, \tag{30}$$

with the positive root τ_p that is called *generalized golden ratio or golden p-ratio*.

If follows from (30) that the generalized golden ratios τ_p are connected by the following fundamental identity:

$$\tau_p^n = \tau_p^{n-1} + \tau_p^{n-p-1} = \tau_p \times \tau_p^{n-1}, \tag{31}$$

where $n = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$.

Note that for the case $p = 0$ Eq. (30) reduces to the trivial equation $x = 2$, with the root $\tau_0 = 2$, and for the case $p = 1$ reduces to the classical golden ratio equation

$$x^2 = x + 1 \tag{32}$$

with the root

$$\tau = \frac{1+\sqrt{5}}{2}, \tag{33}$$

called the *golden ratio or golden proportion* [27,28].

Note that we have obtained the golden ratio (33) and its generalization, the golden p-ratios τ_p, from combinatorial reasoning. It is clear that the generalized Fibonacci p-numbers given by (25) and (26) and the golden p-ratios τ_p, which are the roots of the algebraic equation (30), express some deep mathematical properties of Pascal triangle.

4.3. The generalized principle of the Golden Section

The identity (31) is a source of the identity

$$1 = \tau_p^{-1} + \tau_p^{-(p+1)} = \sum_{i=1}^{\infty} \tau_p^{-(i-1)(p+1)-1}, \tag{34}$$

that expresses the *generalized principle of the Golden Section* [19].

Note that for the case $p = 0$ the identity (34) reduces to the identity:

$$1 = 2^{-1} + 2^{-1} = \sum_{i=1}^{\infty} 2^{-i}, \tag{35}$$

that expresses the "Dichotomy Principle".

For the case $p = 1$ the identity (34) reduces to the identity:

$$1 = \tau^{-1} + \tau^{-2} = \sum_{i=1}^{\infty} \tau^{-(2i-1)}, \tag{36}$$

that expresses the classical "Principle of the Golden Section".

Note that the *Law of structural harmony of systems* developed by the Byelorussian philosopher Eduard Soroko [23] is a brilliant confirmation of the application of the generalized principle of the Golden Section for self-organizing systems.

4.4. The ''golden'' algebraic equations

The ''golden'' algebraic equation (30) has $p+1$ roots $x_1, x_2, x_3, \ldots, x_{p+1}$. It is proved in [22] that the roots $x_1, x_2, x_3, \ldots, x_{p+1}$ have the following properties:

1. If x_k is a root of Eq. (30) where $k = \{1, 2, \ldots, p+1\}$ then we have

$$x_k^n = x_k^{n-1} + x_k^{n-p-1} = x_k \times x_k^{n-1}, \tag{37}$$

where $n = \{0, \pm1, \pm2, \pm3, \ldots\}$.

2. For a given integer $p = 1, 2, 3, \ldots$ and for $k = 1, 2, 3, \ldots, p$, we have

$$x_1^k + x_2^k + x_3^k + x_4 + \cdots + x_p^k + x_{p+1}^k = 1. \tag{38}$$

3. For a given integer $p = 1, 2, 3, \ldots$ the following general algebraic equation:

$$x^n = F_p(n-p+1)x^p + \sum_{t=0}^{p-1}\left[F_p(n-p-t)x^t\right], \tag{39}$$

has the golden p-proportion τ_p as the root. Here $n = p+1, p+2, p+3, \ldots$, and $F_p(n)$ are the Fibonacci p-numbers.

Note that for the case $p = 1$ and $n = 4$ the algebraic equation (39) reduces to the following:

$$x^4 = 3x + 2 \tag{40}$$

that, according to the statement of the Great physicist Richard Feynman, describes the minimal power condition of the butadiene molecule. Feynman expressed his admiration of the golden proportion in the following works: ''What miracles exist in mathematics! According to my theory, the Golden Proportion of the ancient Greeks gives the minimal power condition of the butadiene molecule''.

4.5. Hyperbolic Fibonacci and Lucas functions

In [9,16,18] a very original approach to Binet formulas (12) and (13) let to a new mathematical discovery, *hyperbolic Fibonacci and Lucas functions*, that are a new class of hyperbolic functions (8). The symmetric hyperbolic Fibonacci and Lucas function introduced in [18] have the following analytical expressions:

Symmetrical Fibonacci sine and cosine

$$\mathrm{sFs}(x) = \frac{\tau^x - \tau^{-x}}{\sqrt{5}}, \qquad \mathrm{cFs}(x) = \frac{\tau^x + \tau^{-x}}{\sqrt{5}}, \tag{41}$$

Symmetrical Lucas sine and cosine

$$\mathrm{sLs}(x) = \tau^x - \tau^{-x}, \qquad \mathrm{cLs}(x) = \tau^x + \tau^{-x}. \tag{42}$$

The Fibonacci and Lucas numbers are determined identically through the symmetrical Fibonacci and Lucas functions as the following:

$$F_n = \begin{cases} \mathrm{sFs}(n) & \text{for } n = 2k, \\ \mathrm{cFs}(n) & \text{for } n = 2k+1, \end{cases} \qquad L_n = \begin{cases} \mathrm{cLs}(n) & \text{for } n = 2k, \\ \mathrm{sLs}(n) & \text{for } n = 2k+1. \end{cases} \tag{43}$$

As is noted in [9,16,18], the hyperbolic Fibonacci and Lucas functions, which are extensions of Binet formulas (12) and (13) for continuous domain, transform the Fibonacci numbers theory [27–29] into a ''continuous'' theory because every identity for the hyperbolic Fibonacci and Lucas functions had its discrete analogy in the framework of Fibonacci and Lucas numbers theory. On the other hand, if we take into consideration a great role-played by the hyperbolic functions in geometry, physics and cosmology (''Lobachevsky's hyperbolic geometry'', ''Four-dimensional Minkowsky's world'', etc.), it is possible to expect that the new theory of the new theory of the hyperbolic functions will lead to new results and interpretations in hyperbolic geometry, physics and cosmology.

Note that *Bodnar's geometry* [47] is a brilliant confirmation of the byperbolic Fibonacci Lucas application for simulation of *phyllotaxis phenomenon*.

4.6. A general theory of Binet formulas for Fibonacci and Lucas p-numbers

As is shown in [21], Binet formulas (12) allow the following generalization for the Fibonacci p-numbers. It is proved that for the given $p > 0$ the Fibonacci p-number $F_p(n)$ that is given by the recurrence relation (25) with the initial terms (26) can be represented in the following analytical form:

$$F_p(n) = k_1(x_1)^n + k_2(x_2)^n + \cdots + k_{p+1}(x_{p+1})^n, \tag{44}$$

where $x_1, x_2, \ldots, x_{p+1}$, are the roots of the "golden" algebraic equation (30) and $k_1, k_2, \ldots, k_{p+1}$ are some constant coefficients that are solutions of the following system of algebraic equations:

$$\begin{aligned}
&F_p(0) = k_1 + k_2 + \cdots + k_{p+1} = 0,\\
&F_p(1) = k_1x_1 + k_2x_2 + \cdots + k_{p+1}x_{p+1} = 1,\\
&F_p(2) = k_1(x_1)^2 + k_2(x_2)^2 + \cdots + k_{p+1}(x_{p+1})^3 = 1,\\
&\ldots\\
&F_p(p) = k_1(x_1)^p + k_2(x_2)^p + \cdots + k_{p+1}(x_{p+1})^p = 1,
\end{aligned} \tag{45}$$

where $F_p(0), F_p(1), F_P(2), \ldots, F_p(p)$ are the initial terms of the Fibonacci p-series given by (26).

For the case $p = 1$, the formulas (44) and (45) take the following form:

$$F_1(n) = k_1(\tau)^n + k_2(-1/\tau)^n, \tag{46}$$

$$F_1(0) = k_1 + k_2, \qquad F_1(1) = k_1\tau + k_2(-1/\tau). \tag{47}$$

Solving system (47) we get: $k_1 = \frac{1}{\sqrt{5}}$ and $k_2 = -\frac{1}{\sqrt{5}}$. If we substitute, k_1 and k_2 to (46), we get the well-known Binet formula (12) for the classical Fibonacci numbers.

In [21], a new class of numerical sequences, called *Lucas p-numbers*, was introduced. For a given $p > 0$ the Lucas p-numbers are given by the following analytical formula:

$$L_p(n) = (x_1)^n + (x_2)^n + \cdots + (x_{p+1})^n, \tag{48}$$

where $x_1, x_2, \ldots, x_{p+1}$ are the roots of the algebraic equation (30). It is proved [21] that the formula (48) gives the Lucas p-series $L_p(n)$ ($n = 0, \pm1, \pm2, \pm3, \ldots$), which can be given by the recurrence relation:

$$L_p(n) = L_p(n-1) + L_p(n-p-1), \tag{49}$$

with the initial terms:

$$L_p(0) = p + 1, \qquad L_p(1) = L_p(2) = \ldots = L_p(p) = 1. \tag{50}$$

If we accept $k_1 = k_2 = 1$ in (46) we get the Binet formula for the classical Lucas numbers (9).

Table 5 gives the Lucas p-numbers for the different value of p.

Note that the formulas (44) and (48) give an infinite number of Binet formulas for Fibonacci and Lucas p-numbers. Some useful examples of such formulas for $p = 2, 3, 4$ are given in [21].

It is now difficult to predict in which part of science the above-introduced Binet formulas (44), for the Fibonacci and Lucas p-numbers will have the most effective application. It is clear that the *theory of the Binet formulas* described in [21] is a challenge to the branch of modern mathematics known as the *Fibonacci numbers theory* [27–29], which is now actively developing. The author is sure that the new Binet formulas that are based on combinatorial consideration will attract the attention of theoretical physicists if we take into consideration the active interest of physical science to the Fibonacci numbers and the Golden Section [31–45].

Table 5
Lucas p-numbers

	n												
	0	1	2	3	4	5	6	7	8	9	10	11	12
$L_1(n)$	2	1	3	4	7	11	18	29	47	76	123	199	322
$L_2(n)$	3	1	1	4	5	6	10	15	21	31	46	67	98
$L_3(n)$	4	1	1	1	5	6	7	8	13	19	26	34	47
$L_4(n)$	5	1	1	1	1	6	7	8	9	10	16	23	31
$L_5(n)$	6	1	1	1	1	1	7	8	9	10	11	12	19
$L_6(n)$	7	1	1	1	1	1	1	8	9	10	11	12	13

4.7. Fibonacci matrixes

A theory of Fibonacci matrixes based on Fibonacci p-numbers is stated in [12]. Using the idea of the Q-matrix (16) the following generalization of the Q-matrix was introduced in [12]:

$$Q_p = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{51}$$

Such matrixes were called in [12] *Fibonacci Q_p-matrixes*.

Let us analyze the matrix (51). First of all, we note that all elements of the Q_p-matrix (49) are equal to 0 or 1; here the first column begins from 1 and ends by 1 but all its other elements are equal to 0, the last row of the matrix (51) begins from 1 and all the other elements are equal to 0. The other part of the matrix (51) (without the first column and last row) is an identity $(p \times p)$-matrix. For the cases, $p = 0, 1, 2, 3, 4$ the corresponding Q_p-matrixes have the following form, respectively:

$$Q_0 = (1), \qquad Q_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = Q, \qquad Q_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad Q_4 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that for the case $p = 1$, the Q_1-matrix coincides with the classical Q-matrix (16) [28]. Note also that the Q_p-matrixes have exceptional regularity. For example, the Q_{p-1}-matrix ($p = 1, 2, 3, \ldots$) can be obtained from the Q_p-matrix by means of crossing out last column and the next to the last row in the latter. It means that each Q_p-matrix includes in itself all preceding Q_p-matrixes and is contained into all the next Q_p-matrixes.

In [12] is proved the following useful theorems for the Fibonacci Q_p-matrixes.

Theorem 1. *For a given $p = 0, 1, 2, 3, \ldots$ and $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ we have the following property for the nth power of the Q_p matrix*:

$$Q_p^n = \begin{pmatrix} F_p(n+1) & F_p(n) & \cdots & F_p(n-p+2) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-p) & \cdots & F_p(n-2p+2) & F_p(n-2p+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_p(n-1) & F_p(n-2) & \cdots & F_p(n-p) & F_p(n-p-1) \\ F_p(n) & F_p(n-1) & \cdots & F_p(n-p+1) & F_p(n-p) \end{pmatrix}, \tag{52}$$

where $F_p(n)$ are Fibonacci p-numbers given by (25)–(27).

Theorem 2.

$$\det Q_p^n = (-1)^{pn}, \tag{53}$$

where $p = 0, 1, 2, 3, \ldots$; $n = 0, \pm 1, \pm 2, \pm 3, \ldots$.

It is easy to prove [12] the following remarkable identity for the matrix (52):

$$Q_p^n = Q_p^{n-1} + Q_p^{n-p-1}. \tag{54}$$

Note that for the case $p = 1$ the formula (52) reduces to the formula (17). If we calculate the determinant of the matrix (17) we will get the following well-known identity for the classical Fibonacci numbers:

$$\det Q^n = F(n+1) \times F(n-1) - F^2(n) = (-1)^n. \tag{55}$$

Note that Theorems 1 and 2 are a source of new mathematical results in the field of the Fibonacci number theory. In particular, for the case $p = 2$ it follows from Theorem 2 the following fundamental identity for the Fibonacci 2-numbers.

Theorem 3.

$$\begin{aligned}\det Q_2^n &= F_2(n+1)[F_2(n-2)F_2(n-2) - F_2(n-1)F_2(n-3)] + F_2(n)[F_2(n)F_2(n-3) - F_2(n-1)F_2(n-2)] \\ &\quad + F_2(n-1)[F_2(n-1)F_2(n-1) - F_2(n)F_2(n-2)] = 1.\end{aligned} \tag{56}$$

Note that the identity (59) is a generalization of the similar identity (55) for the classical Fibonacci numbers. It is clear that the identity (56) is only one of the infinite number of the similar identities corresponding to a given p, where $p = 1, 2, 3, \ldots$. It means that Theorems 1 and 2 expand indefinitely the field of Fibonacci's researches.

It is very important to remember that Fibonacci p-numbers are the diagonal sums of Pascal triangle and can be expressed through binomial coefficients according to (27). It means that an infinite number of the identities of the kind (55) and (56), which can be derived from Theorems 1 and 2, in essence, are new unknown properties of Pascal triangle and binomial coefficients. Thus, a theory of the Q_p-matrixes [12] is of interest in combinatorial analysis as it allows to find new properties of Pascal triangle.

4.8. The "golden" matrixes

Let us represent the matrix (17) in the form of the two matrixes given for the even ($n = 2k$) and odd ($n = 2k + 1$) values of the index n:

$$Q^{2k} = \begin{pmatrix} F(2k+1) & F(2k) \\ F(2k) & F(2k-1) \end{pmatrix}, \tag{57}$$

$$Q^{2k+1} = \begin{pmatrix} F(2k+2) & F(2k+1) \\ F(2k+1) & F(2k) \end{pmatrix}. \tag{58}$$

Using the correlations (43) it is possible to represent the matrixes (57) and (58) in the terms of the symmetric hyperbolic Fibonacci functions (41) and (42):

$$Q^{2k} = \begin{pmatrix} \mathrm{cFs}(2k+1) & \mathrm{sFs}(2k) \\ \mathrm{sFs}(2k) & \mathrm{cFs}(2k-1) \end{pmatrix}, \tag{59}$$

$$Q^{2k+1} = \begin{pmatrix} \mathrm{sFs}(2k+2) & \mathrm{cFs}(2k+1) \\ \mathrm{cFs}(2k+1) & \mathrm{sFs}(2k) \end{pmatrix}, \tag{60}$$

where k is the discrete variable, $k = 0, \pm1, \pm2, \pm3, \ldots$

And now we will replace the discrete variable k in the matrixes (59) and (60) by the continuous variable x:

$$Q^{2x} = \begin{pmatrix} \mathrm{cFs}(2x+1) & \mathrm{sFs}(2x) \\ \mathrm{sFs}(2x) & \mathrm{cFs}(2x-1) \end{pmatrix}, \tag{61}$$

$$Q^{2x+1} = \begin{pmatrix} \mathrm{sFs}(2x+2) & \mathrm{cFs}(2x+1) \\ \mathrm{cFs}(2x+1) & \mathrm{sFs}(2x) \end{pmatrix}. \tag{62}$$

It is clear that the matrixes (61) and (62) are a generalization of the Q-matrix (17) for a continuous domain. They have a few unusual mathematical properties. For example, for the case $x = \frac{1}{4}$ the matrix (61) takes the following form:

$$Q^{\frac{1}{2}} = \sqrt{Q} = \begin{pmatrix} \mathrm{cFs}\left(\frac{3}{2}\right) & \mathrm{sFs}\left(\frac{1}{2}\right) \\ \mathrm{sFs}\left(\frac{1}{2}\right) & \mathrm{cFs}\left(-\frac{1}{2}\right) \end{pmatrix}. \tag{63}$$

It is impossible to imagine that it is the "root square from the Q-matrix" but this "Fibonacci's fantasy" follows from the expression (63).

But the most unexpected properties of the matrixes (61) and (62) follow from the following properties of the symmetrical hyperbolic Fibonacci functions [18]:

$$[\mathrm{sFs}(x)]^2 - \mathrm{cFs}(x+1)\,\mathrm{cFs}(x-1) = -1, \tag{64}$$

$$[\mathrm{cFs}(x)]^2 - \mathrm{sFs}(x+1)\,\mathrm{sFs}(x-1) = 1. \tag{65}$$

If we calculate now the determinants of the matrixes (61) and (62) then using (64) and (65) we will come to one more "fantastic" result that is valid for any value of the continuous variable x:

$$\det Q^{2x} = 1, \tag{66}$$

$$\det Q^{2x+1} = -1. \tag{67}$$

4.9. Algorithmic measurement theory

Algorithmic measurement theory is the first author's theory that was stated in the author's books [4,5]. Fibonacci's measurement algorithms [4] are the most unexpected result of this theory. They are based on the following system of standard weights:

$$\{F_p(1), F_p(2), \ldots, F_p(i), \ldots, F_p(n)\}, \tag{68}$$

where $F_p(i)$ is the Fibonacci p-number given by (25) and (26),

Fibonacci's measurement algorithms generate *Fibonacci p-codes:*

$$N = a_n F_p(n) + a_{n-1} F_p(n-1) + \cdots + a_i F_p(i) + \cdots + a_1 F_p(1), \tag{69}$$

where $a_i \in \{0, 1\}$ is the binary numeral of the ith digit of the code (69); n is the digit number of the code (69); $F_p(i)$ is the ith digit weight calculated in accordance with the recurrent relation (25) and (26).

The positional representation of the natural number N in the form (69) has the following abridged notation:

$$N = a_n a_{n-1} \cdots a_1 \cdots a_1. \tag{70}$$

Note that the notion of the Fibonacci p-code include an infinite number of various methods of the binary representations as every number p "generates" its own Fibonacci p-code ($p = 0, 1, 2, 3, \ldots$).

In particular, for the case $p = 0$ the Fibonacci p-code (69) reduces to the classical binary number system:

$$N = a_n 2^{n-1} + a_{n-1} 2^{n-2} + \cdots + a_i 2^{i-1} + \cdots + a_1 2^0. \tag{71}$$

For the case $p = 1$ the Fibonacci p-code (69) reduces to Zeckendorf's representation (14)

For the case $p = \infty$ the Fibonacci p-code reduces to the "unitary" code:

$$N = \underbrace{1 + 1 + \cdots + 1}_{N}. \tag{72}$$

Thus, the Fibonacci p-code given by (69) is a very wide generalization of the binary code (71), Zeckendorf's representation (14) and the "unitary" code (72), are the special cases of the Fibonacci p-code (69).

Fibonacci's measurement algorithms based on (68) are a special case of more general class of optimal measurement algorithms, for which the "effectiveness function" $F_p(n,k)$ ($n = 0, 1, 2, 3, \ldots$; $k = 0, 1, 2, 3, \ldots$) is described by the following recurrence relation [4]:

$$F_p(n,k) = F_p(n; \underbrace{0, 0, \ldots, 0}_{t}, p_{t+1}, p_{t+2,\ldots}, p_k)$$

$$= \sum_{j=0}^{t} F_p(n-1; \underbrace{0, 0, \ldots, 0}_{j}, p_{t+1} - 1, p_{t+2} - 1, \ldots, p_k - 1, \underbrace{p, p, \ldots, p}_{t-j}), \tag{73}$$

with the initial condition:

$$F_p(1; \underbrace{0, 0, \ldots, 0}_{t}, p_{t+1}, p_{t+2}, \ldots, p_k) = t + 1. \tag{74}$$

The recurrence relation (73) with the initial condition (74) contains a number of remarkable formulas of discrete mathematics. Let $p = 0$. It is proved in [4] that for this case, we have

$$F(n,k) = (k+1)^n. \tag{75}$$

Note that this case generates all well-known positional number systems with the radix $k + 1$, in particular, the decimal number system for the case $k = 9$ and the binary number system (71) for the case $k = 1$.

Let us consider the case $p = \infty$. It is proved in [4] that for this case, we have

$$F(n,k) = C_{n+k}^{k} = C_{n+k}^{n}. \tag{76}$$

Note that this case generates an original method of positional number representation based on the binomial coefficients (76).

Let us consider the recurrence formula (73) for the case $k = 1$. It is proved in [4] that for this case the effectiveness function (73) and (74) reduces to the recurrence relation (25) with the initial terms:

$$F_p(0) = 1, F_p(1) = 2, F_p(2) = 3, \ldots, F_p(p) = p + 1. \tag{77}$$

Note that this case generates Fibonacci p-code (69).

The main result of the algorithmic measurement theory together with all unexpected results is demonstrated in Table 6.

Thus, an "unexpectedness" of the main result of the algorithmic measurement theory (Table 6) consists of the following. The general recurrence relation (73) with the initial condition (74) gives in general form an infinite number of the new, at present unknown optimal measurement algorithm. The main recurrence relation (73) and (74) includes a number of the well-known combinatorial formulas as special cases, in particular, the formula $(k + 1)^n$, the formula $C_{n+k}^{k} = C_{n+k}^{n}$ gives the binomial coefficients, the recurrence relation (25) for Fibonacci p-numbers and finally the formulas for the binary (2^n) and natural $(n + 1)$ numbers.

It is clear that the *algorithmic measurement theory* stated in [4,5,7] is of a great interest for different fields of mathematics and general science. First of all, we can consider the recurrence relation (73) as the widest generalization of the recurrence relation (6) for the classical Fibonacci numbers and the recurrence relation (25) for the Fibonacci p-numbers. It means that the algorithmic measurement theory extends infinitely to the field of Fibonacci's research. Secondly, the main results (73) and (74) are of fundamental interest for *combinatorial analysis* because they connect together different fundamental recurrence formulas (for Fibonacci numbers, for binomial coefficients, etc.). Of course, the algorithmic measurement theory is of fundamental interest for *number systems theory* because it includes all well-known positional number systems with natural radices $k + 1$ and generates a number of new positional number systems based on Fibonacci p-numbers, binomial coefficients and so on. These new number systems are a source of new computer projects and concepts, in particular, Fibonacci computer concept [4].

4.10. A new theory of real numbers

A new approach to number theory is stated in [17]. The key idea of this approach is that the theory of numbers depends on a number definition. The *elementary number theory* [24] studies properties of *natural numbers* that have the following definition:

$$N = 1 + 1 + 1 + \cdots + 1 \quad (N \text{ times}), \tag{78}$$

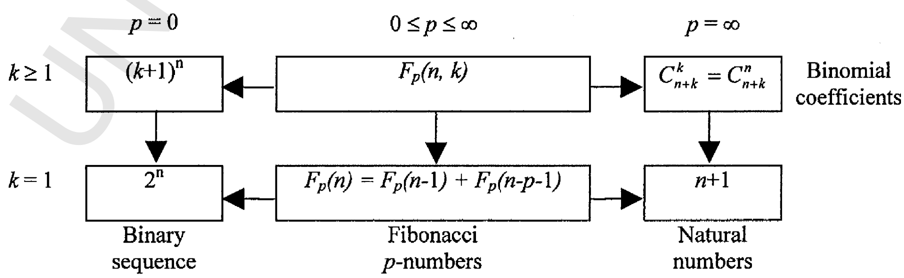
where 1 means geometric segment called a *monad*. Despite limiting simplicity of such a definition, is plays a large role in mathematics and underlies many useful mathematical concepts, for example, concepts of *prime* and *composite* numbers, and also a concept of *divisibility*, one of the main concepts of number theory.

Let us consider now the infinite set of the "standard segments" based on the Golden p-Ratio τ_p:

$$G_p = \{\tau_p^n\}, \tag{79}$$

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$; τ_p^n are the Golden p-Ratio powers connected among themselves by the identity (31).

Table 6



The set (79) generates the following constructive method of the real number A representation called *Code of the Golden p-Proportion*:

$$A = \sum_i a_i \tau_p^i, \tag{80}$$

where $a_i \in \{0, 1\}$ and $i = 0, \pm1, \pm2, \pm3, \ldots$

Note that for the first time a theory of the number systems (80) was developed in the author's book [6] and the author's recent article [17].

Let us consider the special cases of the number system (80). For the case $p = 0$ the formula (80) reduces to the binary number system the underlies modern computers. For the case $p = 1$ the number system (80) reduces to Bergman's number system (15). Finally, let us consider the case $p \to \infty$. For this case it is possible to show, that $\tau_p \to 1$; it means that the positional representation (80) reduces to the definition (78), which underlies the elementary number theory.

Note that for the case $p > 0$ the radix τ_p of the positional number system (80) is an irrational number. It means that, we have came to the *number systems with irrational radices* that are principally a new class of the positional number systems. Bergman's number system (15) is the first number system with irrational radix in the history of mathematics.

Possibly, the number system with irrational radix (15) developed by George Bergman [30] and its generalization given by (80) are the most important mathematical discoveries in the field of number systems after discovery of the positional principle of number representation (Babylon, 2000 BC) and decimal number system (India, fifth century).

Number systems (80), in particular, Bergman's number system (15) originates in a number of non-trivial mathematical results in the number theory. The Z-property of natural numbers proved in [17] is one of such non-trivial properties. The Z-property is given by the following theorem [17].

Theorem 4. *If we represent any natural number N in Bergman's number system* (15) *and then replace here every power of the Golden Ratio* τ^i *in the expression* (15) *by the Fibonacci number* F_i, *where the index i takes its values from the set* $\{0, \pm1, \pm2, \pm3, \ldots\}$, *then the sum arising as a result of such replacing is equal to* 0 *identically independent on the initial natural number N, that is,*

$$\sum_i a_i F_i = 0.$$

4.11. Applications in computer science

Computer science is the main area of the Harmony Mathematics application. These are the following ways of this application:

1. Fibonacci p-codes (69) and Golden Proportion codes (80) are a source of new computer arithmetic and new computer projects described in [4,6,10].
2. The ternary mirror-symmetrical arithmetic described in [14] is a modern original computer invention. This one is a synthesis of Bergman's number system (15) and ternary symmetrical number system used by the Russian engineer and scientist Nikolay Brousentsov in "Setun" computer [46] which is the first in ternary computer history based on the "Brousentsov's Ternary Principle" [14].
3. A new coding theory on Fibonacci and "golden" matrixes [13,19] can be used effectively for redundant coding and cryptographic protection.
4. The "golden" resistive dividers [19] can be used for designing of self-correcting analog-to-digit and digit-to-analog converters.

5. Conclusion and discussion

The main goal of this research is to develop the *Fundamentals of a Harmony Mathematics* that was proclaimed by the author in [11]. Clearly, Harmony concept is very complex subject because it expresses both the quantitative and qualitative aspects of this or that object or phenomenon. However at the mathematical study of the Harmony concept, we disregard the qualitative aspects and focus all our attention on the quantitative aspects of the Harmony concept. Such an approach leads us to the unexpected conclusion, that studying the quantitative aspects of Harmony, that in the Greek language means *combination* and *connection*, can be studied by means of the methods of the combinatorial analysis.

The author developed his research in the following hierarchy:

Binomial theorem, binomial coefficients and Pascal triangle
→ *Fibonacci numbers and the Golden Section*
→ *Fibonacci p-numbers and the Golden p-Sections*
→ *Generalized Principle of the Golden Section*
→ *Binet formulas and Hyperbolic Fibonacci and Lucas functions*
→ *The "golden" algebraic equations based on the Golden p-Sections*
→ *Theory of the Binet formulas for Fibonacci and Lucas p-numbers*
→ *Theory of Fibonacci matrixes following from the Fibonacci p-numbers*
→ *Theory of the "Golden" matrixes following from the hyperbolic Fibonacci functions*
→ *Algorithmic measurement theory*
→ *A new theory of real numbers*
→ *A new computer arithmetic following from Fibonacci p-codes and Golden p-Proportion Codes*
→ *A new coding theory based on the Fibonacci and "golden" matrixes*
→ *Mathematical Theory of Harmony*.

There remains a question: are there in modern science examples of similar theories? Possibly, the mathematical theory of information developed by Claude Shannon [48] is a brilliant example. Really, Information is the same complex concept as Harmony. Developing the theory of the information, Claude Shannon used the concept of *probability* as the initial concept of the theory, and the concept of *entropy* became the basic concept of information theory. It is necessary to emphasize, that Shannon's theory of information is a mathematical theory and can be effectively used for the quantitative analysis of informational systems. In his well-known article "Bandwagon" [48] Shannon warned to be careful in application of this theory to other areas of human activity.

As is known, Shannon's theory of information is sometimes considered as a branch of probability theory. Developing an analogy between Shannon's theory of information [48] and the Harmony Mathematics, it is possible to consider the Harmony Mathematics as a special branch of combinatorial analysis.

From such a point of view it is necessary to approach the question from the practical application side of the Harmony Mathematics. Which are the areas of the effective application of this theory? To answer this question, it is necessary to emphasize, that the most effective areas are those where the quantitative aspects of Harmony are most important. It is possible to name theoretical physics, biology and botany, economy, computer science as examples.

Let us begin with theoretical physics. Over the years many theoretical physicists [33–45] have come to the conclusion, that without the Golden Section it is impossible to develop theoretical physics any further. In this connection, greater hopes in this scientific area are connected with the hyperbolic Fibonacci and Lucas functions [9,16,18]. As is mentioned above, owing to these functions the Golden Section can take a leading role in Lobachevsky's hyperbolic geometry and Minkovsky's geometry. Also the researches of the outstanding physicist Richard Feynman, who founded the "golden" algebraic equation (40) at the research of the power condition of the butadiene particle, are very reassuring. A theory of the "golden" algebraic equations gives an infinite number of various algebraic equations, which possibly model power conditions of other particles.

For biology and botany the new numerical sequences, Fibonacci and Lucas p-numbers are of interest for modeling a cell division [49], the hyperbolic Fibonacci and Lucas functions are of interest for modeling phyllotaxis [47].

As mentioned above, computer science is an area of a wide application of Harmony Mathematics [5–7,10,14].

As for application of Harmony Mathematics in Art works it is necessary to consider it with a sufficient degree of healthy skepticism. In this area of human culture, the qualitative aspect of Harmony is more important than quantitative. Nevertheless, it is possible to expect, that the new Golden Proportions, the Golden p-proportions, can be of interest for proportionality theory.

Acknowledgements

Though all the basic ideas and concepts of the Harmony Mathematics had been developed by the author independently (see author's works [4–17,19]), but in the recent years a significant contribution to development of this theory was made by Boris Rozin (see our common articles [18,20–22]). The author would like to express his gratitude to Boris Rozin for active participation in the development of Harmony Mathematics.

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