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The golden section, secrets of the Egyptian civilization and harmony mathematics

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Abstract

The main goal of the present article is to consider the harmony mathematics from the point of view of the sacral geometry and to show how it can be used in this field. We also consider some secrets of the Egyptian civilization that have relation to the golden section and platonic solids. Briefly, this is considered to be the main concepts involved in harmony mathematics and its application to the sacral geometry.

Religiousness of a scientist is in his enthusiastic worship for laws of harmony.
Albert Einstein

1. Introduction

Some times it seems that a return to the positive mathematical aspects of "sacral geometry" [1] and "sacred knowledge", which are contained in the Talmud, the Bible, the Chinese "Book of Changes", and a reapproachment of religious and scientific outlooks are typical features of certain streams of the present stage in the development of human and scientific culture [2].

Modern science demonstrates a huge number of "sacred knowledge" with applications in different areas. Let us consider a few of them.

The well-known "sacred" number $\tau = \frac{1+\sqrt{5}}{2}$ is used widely in modern science, in particular, in theoretical physics [3–13]. The Russian theoretical physicist Professor Jury Vladimirov from the Theoretical Physics Department of Moscow University recently published the remarkable book "Metaphysics" (2000) [11]. This serious book in theoretical physics concludes with a rather remarkable phrase:

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"*Thus, we can assert that in the theory of weak interactions we find the Golden Section relation, which plays an important role in various spheres of Science and Art*".

"Sacred figures" as platonic solids (Fig. 1) are a source of many scientific ideas and concepts in modern science. As is well known, they were used by Plato in his cosmology. According to this cosmology the four "basic elements" (Fire, Air, Water and Earth) underlie the Universe. According to Plato the atoms of these "elements" have a shape of platonic solids (Fire—tetrahedron in Fig. 1(a), Air—octahedron in Fig. 1(b), Earth—hexahedron or cube in Fig. 1(c), Water—icosahedron in Fig. 1(d)). But Plato considered the dodecahedron in Fig. 1(e) as the main figure of the Universe. It symbolized the whole Universe and the world-wide Mind or Intellect. These ideas were discussed many years later by the great physicist Werner Heisenberg. According to a recent theory the Universe could be a dodecahedron.

According to modern ideas [1], a process of formation of a new life begins with the ovule divided into two cells. Then, on the stage of four cells, an embryo takes the form of the tetrahedron (Fig. 1(a)), then at a stage of eight cells it takes the form of two linked tetrahedrons (a star tetrahedron or a cube). And this process of the embryo development reminds the model of the "life fruit" development, which is based on the "metathron cube" that contains in itself all platonic solids [1].

On November 12 of 1984 in a brief paper published in the very authoritative journal "Physical Review Letters" the experimental evidence of the existence of a metal alloy, which possesses exceptional properties, was presented. The author of the discovery is the Israel engineer Dan Shechtman. The crystal structure of this alloy had the "*icosahedronical*" *symmetry*, i.e. the 5th *order symmetry*, which is strictly forbidden by the classical crystallography. The alloys with such unusual properties were called *quasi-crystals*. Due to this discovery the golden section, which underlies icosahedron (Fig. 1(d)) and the 5th order symmetry (a pentagon), was put in the forefront of modern physics. In the article

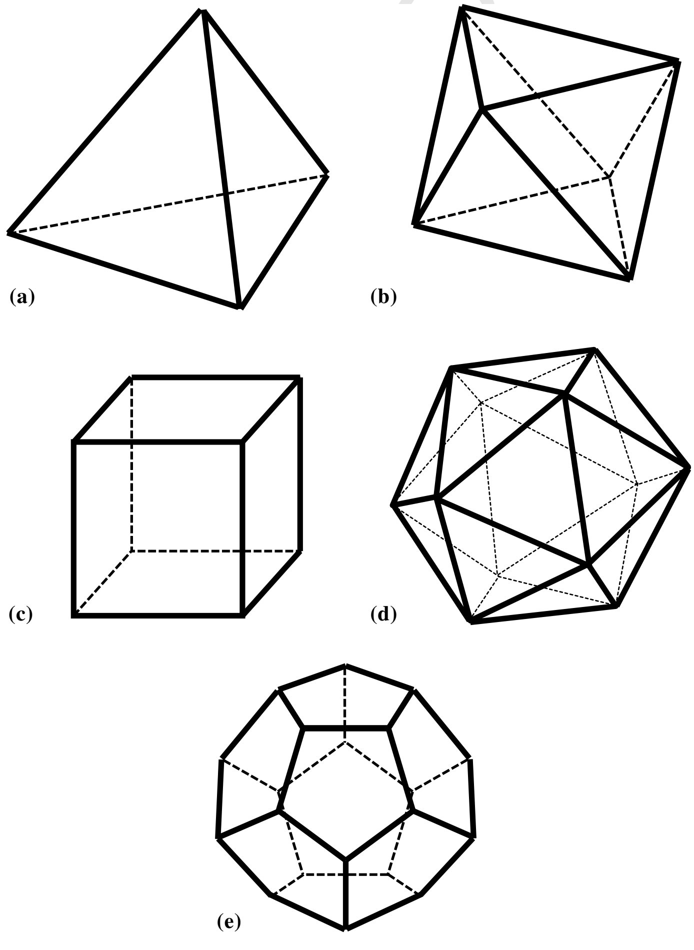

Fig. 1. Platonic Solids.

[14] devoted to this discovery there was stated that "*the importance of the latter as to the world of minerals can be aligned with the advent of the irrational number idea in mathematics*" [14, p. 348].

The Russian scientist Professor Petoukhov found a very interesting application of the Chinese "Book of Changes" in his research on the genetic code. In his remarkable book "Biperiodical table of genetic code and number of protons" (2001) [15] he demonstrated a deep connection between the "table of trigrams" of the "Book of Changes" and the genetic code. These examples are too many to count.

Taking into consideration the great interest of modern science in the golden section and the problem of harmony, the occurrence of a new mathematical direction called *harmony mathematics* is quite natural. The concept of the harmony mathematics was put forward by the author in the lecture "The golden section and modern harmony mathematics" given at the 7th International Conference on Fibonacci numbers and their applications (Austria, Graz, 1996) [16]. In 1998, according to the initiative of the outstanding Ukrainian mathematician academician Mitropolsky, the author gave similar lecture at the meeting of the Ukrainian Mathematical Society (Ukraine, Kiev). According to the scientific recommendation of academician Mitropolsky several important articles on the golden section were published by the author in the Ukrainian academic journals [17–21]. In 2003 the author gave the lecture "A New Kind of Elementary Mathematics and Computer Science based on the Golden Section" at the seminar "Geometry and Physics" of the Theoretical Physics Department of the Moscow University. Later thank to the objectivity of the well-known Egyptian physicist Prof. Mohamed Saledin El Naschie the author published a series of the articles on the harmony mathematics in the International Journal "Chaos, Solitons & Fractals" [22–27].

The main goal of the present article is to show a deep connection between the "sacral geometry" and "harmony mathematics" and the main directions of the harmony mathematics application for the development of the "sacral geometry". A second goal of the present article is to reveal some of the secrets of the Egyptian civilization based on the golden section and platonic solids.

2. The basic geometric relations of the sacral geometry

2.1. What is the sacral geometry?

Let us try to state the fundamental concepts of the sacral geometry following to the remarkable book "sacral geometry" (2000) [1].

As is emphasized in the book [1], sacral geometry is a way to perceive the Universe by Man. Pythagoras was concerned with sacral geometry and considered it as "most concealed science of the Gods". Nature uses widely its applications. And we find examples everywhere, from the "golden" spirals of shells to Fibonacci's arrangements of seeds in sunflower. The term "sacral geometry" is used by archaeologists, anthropologists, philosophers. This term is used to embrace a system of religious, philosophical and spiritual views, which are developed by various cultures during human history.

It is possible to give many examples of sacral geometric applications in the various epochs:

(1) The ancient Greeks attributed the various properties to platonic solids giving to them a special sense. "The god geomertizes",—spoke Plato. As it was mentioned, the cube in Plato's cosmology symbolized the Earth while the dodecahedron and the golden section symbolized the Universe harmony and the "world-wide wisdom". Therefore the building, devoted to the idolized Tsar or Pharaohs, could have the traces of the cube while the temple, devoted to the heavenly god, was constructed on the base of the golden section.
(2) The Christian religion used pectoral cross as its main symbol; in geometrical terms in the middle ages it appeared in the form of the "developed cube".
(3) The ancient Egyptians found out, that the regular polygons can be increased at the constancy of the side ratio by means of addition of strictly marked area named *gnomon*. The Egyptians connected the gnomon concept for a square with the god Osirius who is represented at the Egyptian frescos sedentary on the square throne (a square = to reign). In the basis of the throne the square with the L-shaped gnomon is clearly visible.
(4) The spirals on the Ionia columns of the ancient Greek temples was placed according to the principle of a rotating rectangle that is a method of creation of a logarithmic spiral. It specifies that the ancient Greek architects used in their creations the principles of sacral geometry.

One of the most amazing ideas, which penetrate the sacral doctrines of all civilizations, is that the Universe exists as harmonious and proportional whole. The harmony is a basis of the beautiful. The Egyptian goddess Maat represented by itself an embodiment of a principle of the natural order of things, proportional measure and balance as the eternal truth of nature. The Greeks, whose teachers were Egyptians, connected the word "Cosmos" that expresses harmony and beauty with the Universe.

2.2. The five basic geometric relations of the sacral geometry

There is a group of the five basic relations of the sacral geometry, which can be found all over the world starting from Japanese temples to the Great Pyramid [1]. Let us consider these relations:

(1) *The number* $\pi = 3.14$. *The number* π is the main proportion of a circle and sphere. In the sacral geometry the circle represents spiritual empires. At the Earth there are many sacred places in the form of a circle, which are used for carrying out spiritual rituals. The labyrinth on the floor of the cathedral in Shartre in France (Fig. 2) is one of the best known among them.

(2) *The square root of the number 2*: $\sqrt{2} = 1.414$. The square root of the number 2 is connected with such "sacred" figure as a *square* (Fig. 3). Let us remind ourselves that the faces of the cube, the platonic solids, which symbolized the Earth in Plato's cosmology, have a square shape.

The square was found out in one of the most sacred places, the *Solomon temple*. There the ark of the Precept and others most valuable things of the jew were stored. In the sacral geometry the mathematical root becomes like to a vegetative root, both roots represent by themselves a principle of growth and transformation. The vegetative root is in the ground. It breaks (divides) the dense mineral elements of the ground, which then are transformed by the root to a tissue of a plant. The mathematical root is enclosed in a polygon (a square). It leads to the incommensurable (irrational) relations, which cannot be represented in a rational form. The discovery of the *incommensurable line segments*, which led to the development of the theory of irrationalities and irrational numbers and finally to the creation of modern "continuous" mathematics, is connected with the number $\sqrt{2}$.

(3) *The square root of the number 3 and Vesica piscis*: $\sqrt{3} = 1.732$. A crossing of the two circles, when each circle passes through the center of another one, forms the geometrical figure named *Vesica piscis* ("the fish bubble") (Fig. 4). Its spiritual significance was highly valued by the Renaissance artists who used it widely in painting and architecture.

The amazing properties of the *Vesica piscis* were a subject of a deep study in the sacral geometry. The equilateral triangle, which is formed by points A, B и C (Fig. 4), is one of the earliest mystical symbols known to mankind.

The equilateral triangle, which is formed as a result of the two circles crossing (Fig. 4), represents a base, on which the sacral geometry was constructed. Let us remember that the faces of such important platonic solids as *tetrahedron, octahedron* and *icosahedron* have a shape of the equilateral triangle. Like the square, which is expressed by the number $\sqrt{2}$, the equilateral triangle is expressed by the number $\sqrt{3}$, which has a deep metaphys-

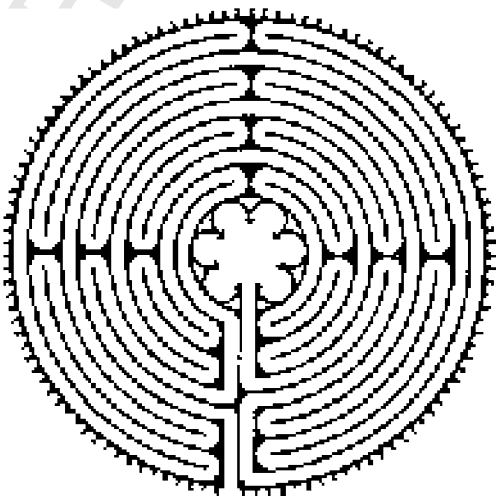

Fig. 2. The labyrinth in the Shartre cathedral.

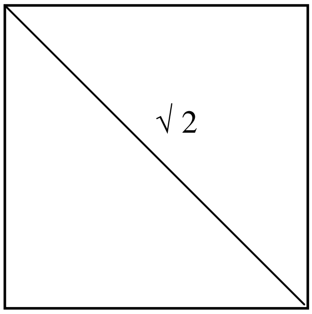

Fig. 3. The square.

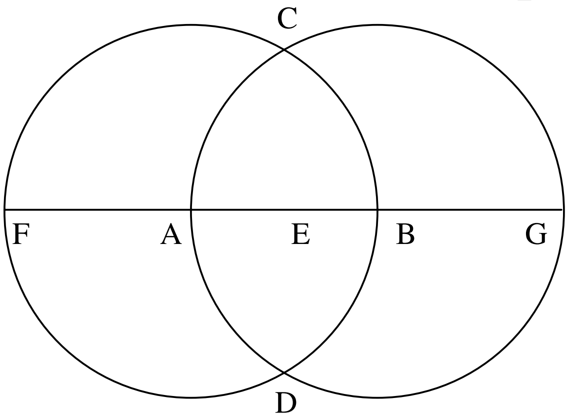

Fig. 4. *Vesica piscis*.

ical meaning. The equilateral triangle possesses the best radiating properties. The form of this triangle defines its fine qualities as generator of radiant energy on greater distances. The great Russian scientist Tsiolkovsky put forward an idea to cut down in the Siberian taiga a gigantic equilateral triangle in order to establish a contact to extraterrestrial civilizations.

(4) *The square root of the number 5*: $\sqrt{5} = 2.236$. This number appears at the research of such sacred figure as a *double square* (Fig. 5). Pythagoreans esteemed the number five as sacred. The double square can be found in the most sacred places: from the Pharaoh chamber in the Great Pyramid up to King Solomon temple.

(5) *The number of PHI*: $\Phi = \frac{1+\sqrt{5}}{2} = 1.618$ (the golden section). The golden section presents by itself a "primary beauty"; that any thing, which is beautiful and pleasing to the eye, should conform to the principle of the golden section. The designation Φ originates from the first letter of the name of the Greek sculptor Phidius who used widely the principle of the golden section in his famous sculptures.

Most clearly the principle of the golden section can be seen in the pentagon (Figs. 5 and 6), a regular polygon, from which the *pentagram* or *pentacle* follow. Let us remember that the faces of the *dodecahedron*, which symbolized the Universe and the world-wide wisdom in Plato's cosmology, have a pentagonal shape. As it was mentioned repeatedly, the pentagram was considered by Pythagoreans as the sacred figure and the main symbol of their sacred union.

Thus, there are the five mysterious geometric relations, namely, the numbers π, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and $\Phi = \frac{1+\sqrt{5}}{2}$ (the golden section). They are present at all sacred places on the Earth and form the basis of all sacral–geometrical constructions. These symbols of the sacral geometry transfer universal principles to understanding. They show the patterns of the creator's basic laws, which proceed from the world-wide mind or intellect. The wisdom of nature offers the best example of how it is best to live. This wisdom is supposed to be transferred to us through the symbols of the sacral geometry.

As it is stated in [1] the sacral geometry uses widely a number of numerical series, in particular, Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ and binomial coefficients that form very important mathematical object called *Pascal triangle* (Fig. 7).

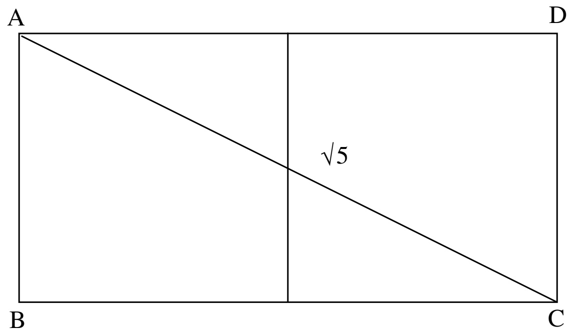

Fig. 5. The double square.

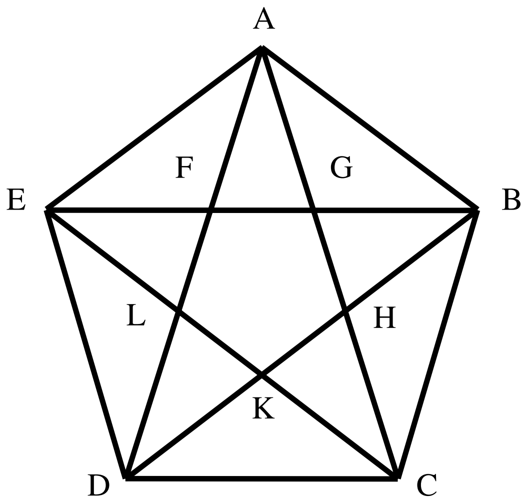

Fig. 6. The regular pentagon and pentagram (pentacle).

									1									
								1		1								
							1		2		1							
						1		3		3		1						
					1		4		6		4		1					
				1		5		10		10		5		1				
			1		6		15		20		15		6		1			
		1		7		21		35		35		21		7		1		
	1		8		28		56		70		56		28		8		1	
1		9		36		84		126		126		84		36		9		1

Fig. 7. Pascal triangle.

Pascal triangle symbolizes a process of sell division because the sum of the Pascal triangle elements is realized by doubling of the preceding sum. Such character of the numbers doubling simulates a simple division of biological cells and other biological organisms during reproduction. Each cell as a result of the division turns into two cells, which turn into four cells and so on. In nonlinear dynamics this may be called *Feigenbaum doubling*.

3. The secrets of the Egyptian civilization

3.1. Phenomenon of the ancient Egypt

In the beginning of the 20th century in Saqqara (Egypt) archaeologists opened the crypt, in which the remnants of an Egyptian architect by name of Khesi-Ra were buried. In the literature this name is often written as Khesira. It is supposed that Khesi-Ra was a contemporary of Imhotep who lived in the period of the Pharaoh Zoser (27th century BC) because in the crypt the printings of this Pharaoh were found.

The wood boards–panels, which were covered with a magnificent thread, were extracted from the crypt alongside with the different material assets. In total, there were 11 boards in the crypt; among them only the five boards were preserved; the remaining panels are completely broken down because of moisture reached to the crypt. On all preserved panels architect Khesi-Ra is figured. He is surrounded with different figures having symbolical significance (Fig. 8). For long time the assigning of the panels from Khesi-Ra's crypt was vague. At first the Egyptologists accepted these panels as false doors. However, since the 60th of the 20th century the situation with the panels began to become clearer. The Russian architect Shevelev, paid particular attention to one of the panels, where the staffs in Khesi-Ra's hands are proportioned as $1:\sqrt{5}$, that is, as the ratio of the side to the diagonal of the "double square" (Fig. 5). This observation becomes the starting point for the research of other Russian architect Shmelev who made careful geometrical analysis of Khesi-Ra's panels and as a result came to the sensational discovery described in the brochure "*Phenomenon of ancient Egypt*" (1993) [28].

Let us give the word to the author of the discovery to speak for himself:

"*But now, after the comprehensive analysis by the method of proportions we have good causes to assert that Khesi-Ra*'*s panels are the Harmony rules encoded by geometric language ... So, in our hands we have the concrete material evidences, which shows us in "plain text" the highest level of abstract thought of the Ancient Egypt intellectuals. The artist, who cut the panels with amazing accuracy, jeweler refinement and masterly ingenuity, demonstrated the rule of the "Golden Section" in its broadest range of variations. The outcome is the "GOLDEN SYMPHONY" presented by the ensemble of the highly artistic works, which testifies not only for ingenious talents of their creator, but also verifies convincingly that the author was aware the Harmony secrets. This genius was of the "Golden Business Craftsman" with the name of Khesi-Ra*".

But who was Khesi-Ra? The ancient text tells us that Khesi-Ra was a chief of Destius, a chief of doctors, a writer of the Pharaoh, a priest of Gor, a main architect of the Pharaoh, a supreme chief of South Tens, and a carver. Analyzing the listed above Khesi-Ra's regalia, Shmelev paid particular attention to the fact that Khesi-Ra was the priest of Gor. In the ancient Egypt Gor was considered as the god of harmony and therefore to be the priest of Gor means to execute the functions of the harmony keeper. As follows from his name, Khesi-Ra was elevated to the rank of the god of Ra (God of the Sun). Shmelev guessed that Khesi-Ra could get this high award for "*the development of aesthetic ... principles of*

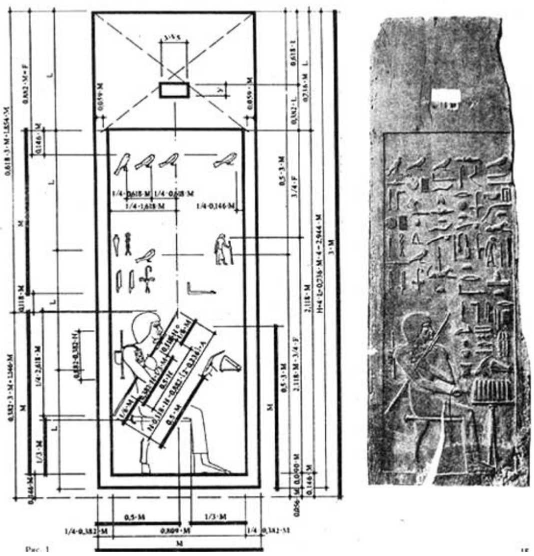

Fig. 8. Khesi-Ra's panel.

the canon system that reflects the harmonious fundamentals of the Universe ... The orientation on the harmony principle gave for the ancient Egypt civilization the path to unprecedented flowering of culture; this flowering falls to the period of Zoser-Pharaoh, when the system of written signs was created. Therefore it is possible, that the Zoser-pyramid became the first experimental pyramid. Then according to the program, which was elaborated under Khesi-Ra's supervision, followed the building of the unified complex of the Great Pyramids in Giza.''

And in conclusion we consider one more quotation from Shmelev's brochure [28]:

''*It is only necessary to recognize that the ancient Egypt civilization was the super-civilization, which was explored by us extremely superficially, and this fact demands a qualitatively new approach to the study of its richest heritage .... The outcomes of the Khesi-Ra's panels research demonstrate that the sources of modern science and culture are in the boundless historical layers, which feed a creativity of the craftsmen of our days with great ideas. These ideas inspired for long time the rushes of the outstanding mankind representatives. And our purpose is not lose a unity of the linking thread.*''

3.2. The great pyramid

Among the gigantic Egyptian pyramids the great pyramid of the Pharaoh Khufu is of special interest. Before to begin the analysis of the form and sizes of Khufu's pyramid it is necessary to remember the Egyptian measure system. The ancient Egyptians used three measuring units: ''elbow'' (466 mm) equaling to 7 ''palms'' (66.5 mm), which, in turn, was equal to 4 ''fingers'' (16.6 mm).

Let us analyze the sizes of Khufu's pyramid (Fig. 9) following the remarkable book of the Ukrainian scientist Nickolai Vasutinskiy [29].

The majority of the researchers believe that the length of the side of the Pyramid basis, for example, GF is equal to $L = 233.16$ m. This value corresponds almost precisely to the 500 ''elbows''. We will have a full comprehension of the 500 ''elbows'', if we consider that the length of the ''elbow'' is equal to 0.4663 m. The height of the pyramid (H) is estimated by the researchers variously from 146.6 m up to 148.2 m. And in dependence on the adopted pyramid height, all proportions of its geometric elements will change considerably. In what is the cause of distinctions in the estimation of the pyramid height? Strictly speaking, Khufu's pyramid is truncated. Its topic platform today has the size approximately to 10×10 m, but one century back it was equal to 6×6 m. Apparently, that the top of the pyramid was dismantled and now it does not fit to the initial pyramid.

Estimating the pyramid height, it is necessary to take into consideration such physical factor, as ''shrinkage'' of construction. For long time under the effect of the enormous pressure, which reaches 500 ton on 1 M^2 of the undersurface, the pyramid height decreased in comparison to its initial height.

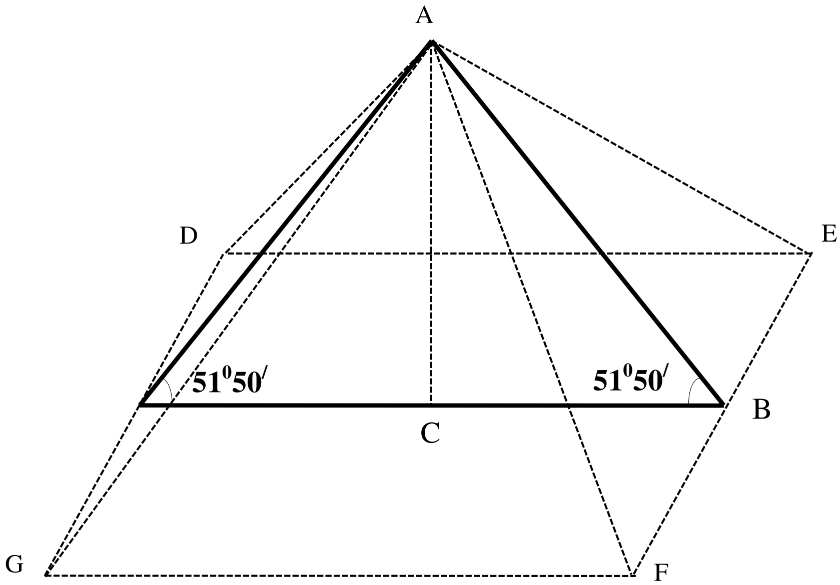

Fig. 9. The geometric model of Khufu's Pyramid.

What was the initial height of the pyramid? This height can be reconstructed if we find the main "geometrical idea" of the pyramid.

In 1837 the English colonel G. Vaise measured the inclination angle of the pyramid faces: it appeared equal $\alpha = 51°51'$. The majority of the researchers recognize this value and today. The indicated value of the inclination angle corresponds to the tangent equal to 1.27306. This value corresponds to the ratio of the pyramid height AC to the half of its basis CB (Fig. 9), that is, $AC/CB = H/(L/2) = 2H/L$.

And here the researchers expected a big surprise! If we take the square root from the golden proportion $\sqrt{\tau}$, we will get the following outcome $\sqrt{\tau} = 1.272$. Comparing this value with the value $\mathrm{tg}\alpha = 1.27306$, we can see that these values are very close among themselves. If to take the angle $\alpha = 51°50'$, that is, to decrease it by one arc minute the value of $\mathrm{tg}\alpha$ will become equal to 1.272, that is, will be equal exactly to the value of $\sqrt{\tau}$. It is necessary to note, that in 1840 G. Vaise repeated his measurements and corrected the value of the angle to $\alpha = 51°50'$.

These measurements resulted in the following rather interesting hypothesis: the ratio $AC/CB = \sqrt{\tau} = 1.272$ was put in the basis of the triangle ACB of the Khufu's pyramid! Let us consider now the right triangle ACB with the sides:

$$CB = 1; \quad AC = \sqrt{\tau}.$$

According to Pythagoras theorem we can calculate the length of AB as follows:

$$AB = \sqrt{CB^2 + AC^2} = \sqrt{1+\tau} = \sqrt{\tau^2} = \tau.$$

It means that the right triangle ACB, which underlies Khufu's pyramid, is based on the golden proportion τ!

Then if we use the hypothesis, that the "golden" right triangle ACB is the main "geometrical idea" of Khufu's pyramid, from here it is easy to calculate the "designed" height of Khufu's pyramid. It is equal:

$$H = (L/2) \times \sqrt{\tau} = 148.28 \text{ M}.$$

Let us derive now some other relations for Khufu's pyramid that result from the "golden" hypothesis. In particular, let us find the ratio of the external area of the pyramid to the area of its basis. For this purpose we take the length of the leg CB for the unit, that is: $CB = 1$. But then the length of the side of the pyramid basis $GF = 2$, and the area of the basis EFGH will be equal to $S_{\mathrm{EFGH}} = 4$. Let us calculate now the area of the lateral face of Khufu's pyramid. As the height AB of the triangle AEF is equal to τ, the area of each lateral face will be equal to $S_\Delta = \tau$. Then the common area of all four lateral faces of the pyramid will be equal to 4τ and the ratio of the summary external area of the pyramid to the area of its basis will be equal to the golden proportion! This also is the main geometrical secret of Khufu's pyramid!

3.3. The mystery of the Egyptian calendar

The Egyptian calendar that was designed in the 4th millennium BC was one of the first solar calendars. In this calendar the year consisted of 365 days. The year was divided into 12 months equaling for 30 days in each; at the end of the year the 5 holidays, which did not entered to the month days, were added. Thus, the Egyptian calendar year had the following structure: $365 = 12 \times 30 + 5$. Note that the Egyptian calendar is the prototype of the modern calendar.

There is a question: Why did the Egyptians sectioned the calendar year into 12 months? Let us remind ourselves that there were calendars with other number of months in one year. For example, in the Maya calendar the year consisted of 18 months for 20 days in one month. The next question is the following: Why each month had 30 days in the Egyptian calendar? One may put some questions concerning to the Egyptian system of time measurement, in particular concerning to the choice of such units of time, as *hour*, *minute*, *second*. In particular, there is a question: Why 1 h was chosen so, that it was packed exactly 24 times in 1 day, that is, why 1 day = 24 (2×12) h? Further: Why 1 h = 60 min, and 1 min = 60 s? The same questions concern to a choice of the angular values units, in particular: Why a circumference is divided into 360°, that is, why $2\pi = 360° = 12 \times 30°$? We can add to these questions and other ones, in particular: Why the astronomers recognized expedient to consider that there are 12 "zodiacal" signs, though actually during the motion on the ecliptic the Sun intersects 13 cancellations? And more one "strange" question: why the Babylonian number system had a rather exotic basis, the number 60?

Analyzing these questions, we can find that here with a surprising constancy the four numbers are repeated: 12, 30, 60 and the number 360, which is derivative from them: $360 = 12 \times 30$. There is a question: is there possibly any scientific idea, which could give simple and logical explanation of the usage of these numbers in the Egyptian calendar, their system of time and angle measurement?

To answer this question we return once again to the dodecahedron in Fig. 1(e), which is based on the golden section. It is known that the dodecahedron has 12 faces, 30 edges and 60 plane angles on its surface. Let us compare now these numbers with the cycles of the solar system. The solar system has three basic cycles that are determined by the movement of the Earth, Saturn and Jupiter around the Sun: the 12 year's Jupiter cycle, the 30 year's Saturn cycle, at last, the 60 year's main cycle of the solar system. Thus, between such perfect spatial figure, as dodecahedron, and the solar system there is a deep mathematical connection. Due to this unexpected correlation between the numerical characteristics of dodecahedron and the cyclic processes in the solar system the antique scientists decided to choose dodecahedron as the "main figure of the Universe", which symbolized the Universe harmony and the world-wide intellect! And then the Egyptians decided that all their main systems (calendar system, systems of time and angle measurement) should be subordinated to the numerical parameters of dodecahedron! As according to the ancient ideas the motion of the Sun on the ecliptic had strictly circumferential nature, they divided the ecliptic by the 12 zodiac signs with the distance between the adjacent zodiac signs in 30°. Due such genius decision the Egyptians coordinated the yearly motion of the Sun by the ecliptic with the structure of their calendar year: *one month corresponded to the movement of the Sun on the ecliptic between the two adjacent zodiac signs*! Moreover, the movement of the Sun by the ecliptic on 1° corresponded to 1 day in the Egyptian calendar year! Thus, the ecliptic automatically was divided by 360°. Then, using the number 12, the Egyptians divided each half of 1 day into the 12 parts (the 12 faces of dodecahedron) and introduced 1 h, the major unit of time. By dividing 1 h into 60 min (60 plane angle on the surface of dodecahedron), the Egyptians in such way introduced 1 min, the next relevant unit of time. Precisely also they introduced 1 s, the most small-sized for that period unit of time (1 min = 60 s).

Thus, by selecting dodecahedron as the main "harmonious" figure of the Universe and strictly following to its numerical characteristics 12, 30 and 60, the Egyptians constructed the extremely ordered calendar, and also the systems of time and angle measurement. These systems were completely coordinated with their "theory of harmony" based on the golden proportion because this proportion underlies dodecahedron.

Here such surprising conclusions follow from comparison of dodecahedron with the solar system. And if our hypothesis is correct (let somebody attempts to deny it), it follows from here that a lot of millennia the mankind lives according to the golden proportion! And each time, when we look at the index dial of our watch, which also is constructed by the use of the numerical characteristics of dodecahedron 12, 30 and 60, we touch to the "main secret of the Universe", the golden proportion!

4. Harmony mathematics and sacral geometry

4.1. The main mathematical concepts and theories of the harmony mathematics

As was mentioned, for the first time a concept of the harmony mathematics was introduced by the author in the lecture "The Golden Section and Modern Harmony Mathematics" given at the 7th International Conference on Fibonacci numbers and their applications (Austria, Graz, 1996) [16]. However, thank to the support of the outstanding Ukrainian mathematician–academician Mitropolsky and the famous Egyptian physicist Prof. El Naschie, the author published a series of the articles on the harmony mathematics in the Ukrainian academic journals and in the International journal "Chaos, Solitons and Fractals" [17–27]. That is why, academician Mitropolsky and Professor El Naschie can be named as the "Godfathers" of the new mathematics, the harmony mathematics.

In the article "Fundamentals of a new kind of Mathematics based on the Golden Section" [27] the author gave the following hierarchy of the main mathematical concepts and theories of the harmony mathematics:

Binomial theorem, binomial coefficients and Pascal triangle →
Fibonacci numbers and the Golden Section →
Fibonacci p-numbers and the Golden p-Sections →
Generalized Principle of the Golden Section →
Binet formulas and Hyperbolic Fibonacci and Lucas functions→
The "golden" algebraic equations based on the Golden p-Sections→
Theory of the Binet formulas for Fibonacci and Lucas p-numbers→
Theory of Fibonacci matrices following from the Fibonacci p-numbers →
Theory of the "Golden" matrices following from the hyperbolic Fibonacci functions →
Algorithmic measurement theory → A new theory of real numbers →
A new computer arithmetic following from Fibonacci p-codes and Golden p-Proportion Codes →
A new coding theory based on the Fibonacci and "golden" matrices→
Mathematical Theory of Harmony

Thus, the harmony mathematics is a wide extension and generalization of the *Fibonacci numbers theory* [30–32] that was developed during many centuries by the Italian 13th century mathematician Fibonacci, by the famous Italian 16th century mathematician Luca Pacioli, who was a friend and scientific advisor of Leonardo da Vinci, by the French 19th century mathematicians Binet and Lucas and in the last decades by the American Fibonacci mathematicians published their articles in "The Fibonacci quarterly". The harmony mathematics has two important consequences for the classical Fibonacci numbers theory:

(1) In the first place, due to the introduction of the concepts of the *generalized Fibonacci and Lucas numbers* (the Fibonacci and Lucas p-numbers) [33–36,25] and the *generalized golden sections* (the golden p-sections) [33] and also due to the introduction of the *generalized "golden" algebraic equations* [26], the *generalized Fibonacci matrixes* (the Q_p-matrixes) [20], the *generalized Binet formulas for the Fibonacci and Lucas p-numbers*[25], the subject of Fibonacci's researches broadens infinitely. Note that the number p takes here its values from the set $\{0, 1, 2, 3, \ldots\}$.
(2) In the second place, due to the introduction of the hyperbolic Fibonacci and Lucas functions [18,22,37] the classical Fibonacci numbers theory is degenerated because it becomes as the special ("discrete") case of the more general ("continuous") theory of the hyperbolic Fibonacci and Lucas functions [18,22,37]. Note that due to the hyperbolic Fibonacci and Lucas functions the golden section, Fibonacci and Lucas numbers become the heart of the hyperbolic geometry and Einstein's theory of relativity, that is, the Fibonacci numbers theory passes on a new stage of its development.

4.2. The golden p-rectangles

However, the new mathematical concepts and theories of the harmony mathematics are of fundamental interest for the sacral geometry. As example, let us consider so-called *golden p-rectangle* [21], in which the ratio of the sides is equal: τ_p:1, where τ_p is the golden p-proportion, the root of the "golden" algebraic equation:

$$x^{p+1} - x^p - 1 = 0 \tag{1}$$

where $p = 0, 1, 2, 3, \ldots$.

Let us remember ourselves that the golden p-proportions τ_p is a new class of irrational numbers that express some deep mathematical properties of Pascal triangle. The golden 0-proportion is equal to the number 2; all rest golden p-proportions are irrational numbers that are between 2 and 1. Below we can see the table of the golden p-proportions τ_p and inverse to them numbers $\beta_p = \frac{1}{\tau_p}$ that correspond to the initial values of the index p (Table 1).

Let us consider special cases of the golden rectangle with ratio: τ_p:1. As follows from Table 1 for the case $p = 0$ the golden 0-proportion is equal to $\tau_0 = 2$. For this case the golden p-rectangle is reduced to the rectangle with the side ratio: 2:1. But this rectangle coincides with the "double square" (Fig. 5) that is used widely in the sacral geometry and has relation to the "sacred" number $\sqrt{5}$. For the case $p \to \infty$ the golden p-proportion $\tau_p \to 1$. It means that for this case the golden p-rectangle degenerates into the square (Fig. 3) that has a relation to the "sacred" number $\sqrt{2}$. At least, for the case $p = 1$ the golden 1-proportion $\tau_1 = \tau = \frac{1+\sqrt{5}}{2} = 1.618$, that is, the golden p-rectangle is reduced to the well-known golden rectangle.

Thus, the golden p-rectangle concept includes in itself, at least, three sacred figures (the square, the "double square" and the golden rectangle) that have a direct relation to 3 of 5 basic geometric relations of the sacral geometry, $\sqrt{2}$, $\sqrt{5}$ and $\Phi = \frac{1+\sqrt{5}}{2}$ (the golden proportion). It means that the golden p-proportions, possibly, are a new class of the "sacred" irrational numbers and the golden p-rectangle concept, which generates an infinite number of new harmonious rectangles, possibly, has a fundamental importance for the sacral geometry. One may except that the golden 2-rectangle with the ratio 1.465:1, the golden 3-rectangle with the ratio 1.380:1 and other can be used in the sacral geometry.

Table 1
The golden p-proportions

p	0	1	2	3	4	5	6	…	∞
τ_p	2	1.618	1.465	1.380	1.324	1.285	1.255	…	1
β_p	0.5	0.618	0.682	0.724	0.755	0.778	0.796	…	1

4.3. The generalized principle of the golden section

In [23] the generalized principle of the golden section was introduced. This one is expressed by the following correlation:

$$1 = \tau_p^{-1} + \tau_p^{-(p+1)} = \sum_{i=1}^{\infty} \tau_p^{-(i-1)(p+1)-1}, \tag{2}$$

where τ_p is the golden p-proportion given in Table 1.

For the case $p = 0$ the expression (2) is reduced to the following identity:

$$1 = 2^{-1} + 2^{-1} = \sum_{i=1}^{\infty} 2^{-i}, \tag{3}$$

that expresses the "dichotomy principle".

For the case $p = 1$ the identity (2) is reduced to the identity

$$1 = \tau^{-1} + \tau^{-2} = \sum_{i=1}^{\infty} \tau^{-(2i-1)}, \tag{4}$$

that expresses the classical "principle of the golden section".

As the "dichotomy principle" (3) and the classical "principle of the golden section" are used widely in the sacral geometry [38], one may expect that the generalized principle of the golden section can be used in the sacral geometry too.

4.4. Soroko's law of structural harmony of systems

Recently the Byelorussian philosopher Eduardo Soroko developed a very original approach to the structural harmony of systems [40]. For this purpose he used the generalized golden proportions, which are given in Table 1, and formulated his famous "Law of Structural Harmony of Systems" as follows:

"*The Generalized Golden Proportions are invariants, which allow for natural systems in process of their self-organization to find harmonious structure, stationary regime of their existence, structural and functional stability*".

According to Soroko's opinion [40], this general law has the best applications in nature, science and art for modeling of the processes in self-organizing systems (substance structure, technology of structural-complex products, ontogenetic structures of organism, human intellect and its creations). What is a principal peculiarity of Soroko's Law for modern science and the sacral geometry? Starting from Pythagoras the harmony researchers connected the harmony concept with the classical golden proportion $\tau = \frac{1+\sqrt{5}}{2} = 1.618$ that was considered as the only and unique proportion of harmonious systems. Soroko's Law asserts that there are an infinite number of the "harmonious" states of one and the same systems corresponding to the numbers τ_p or to the inverse to them numbers $\beta_p = \frac{1}{\tau_p}$ given in Table 1. Note that Soroko's Law extends considerably god's chances to construct the "harmonious Universe". It means that Soroko's Law is of fundamental interest for the sacral geometry!

4.5. The "asymmetry principle" of the living nature

In the sacral geometry the "dichotomy principle" (3) was used for modelling of sell division [1]. However, the newest biological researches [39] show that a process of sell division is more complex process, which is subordinated to the "asymmetry principle" of the living nature stated in [23]. According to the "asymmetry principle" mathematically a process of sell division is described by the Fibonacci p-numbers given by the following recurrence relation:

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \quad \text{for } n > p+1; \tag{5}$$

with the initial terms

$$F_p(0) = 0, \quad F_p(1) = F_p(2) = \cdots = F_p(p) = 1, \tag{6}$$

where $p = 0, 1, 2, 3, \ldots$.

Note that the recurrence relation (5) and (6) generates an infinite number of different recurrence sequences given in Table 2.

Table 2
Fibonacci p-numbers

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$F_0(n)$	0	1	2	4	8	16	32	64	128	512	1024	2048	4096
$F_1(n)$	0	1	1	2	3	5	8	13	21	34	55	89	143
$F_2(n)$	0	1	1	1	2	3	4	6	9	13	19	28	41
$F_3(n)$	0	1	1	1	1	2	3	4	5	7	10	14	19
$F_4(n)$	0	1	1	1	1	1	2	3	4	5	6	8	11

Table 3
Lucas p-numbers

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$L_0(n)$	1	1	2	4	8	16	32	64	128	512	1024	2048	4096
$L_1(n)$	2	1	3	4	7	11	18	29	47	76	123	199	322
$L_2(n)$	3	1	1	4	5	6	10	15	21	31	46	67	98
$L_3(n)$	4	1	1	1	5	6	7	8	13	19	26	34	47
$L_4(n)$	5	1	1	1	1	6	7	8	9	10	16	23	31

In [25] the *generalized Lucas numbers* called *Lucas p-numbers* were introduced. They are given by the following recurrence relation:

$$L_p(n) = L_p(n-1) + L_p(n-d-1), \tag{7}$$

with the initial terms:

$$L_p(0) = d+1, \quad L_p(1) = L_p(2) = \cdots = L_p(d) = 1. \tag{8}$$

Note that the recurrence relation (7) and (8) generates an infinite number of different recurrence sequences given in Table 3.

Let us remember ourselves that the Fibonacci and Lucas p-numbers have direct relation to the golden p-proportions τ_p because the ratios of the adjacent Fibonacci p-numbers $F_p(n)/F_p(n-1)$ and the adjacent Lucas p-numbers $L_p(n)/L_p(n-1)$ aim for τ_p for the case $n \to \infty$.

One may expect that a new class of irrational numbers given in Table 1 and new classes of the recurrence sequences given in Tables 2 and 3 could be of big interest for the sacral geometry and theoretical physics.

4.6. The "hyperbolic world" that is based on the golden section

Starting since the 19th century, after the discovery of hyperbolic geometry by the Russian geometer Nickolay Lobachevsky, the hyperbolic ideas got wide spreading in modern science, in particular, in physics and biology. In this connection, a new class of hyperbolic functions based on the golden section and called *hyperbolic Fibonacci and Lucas functions* [18,22,37] is fundamental for the study of the sacral geometry and modern science. The idea of the creation of a new theory of the "hyperbolic world" based on the hyperbolic Fibonacci and Lucas functions may become the "key idea" of the sacral geometry and theoretical physics. A development of this idea can become a source of many useful and fruitful concepts and theories that can confirm the surprising prediction of Pythagoras, Plato, Leonardo da Vinci, Luca Pacoli, Johannes Kepler, Mohamed El-Naschie and Vladimirov that the "physical world", which surrounds us, is based on the golden section! In this connection, the research of the Ukrainian architect Oleg Bodnar on the creation of the new geometric phyllotaxis theory [41], which is based on the hyperbolic Fibonacci and Lucas functions, is of fundamental interest for the sacral geometry and natural sciences.

5. Conclusion

Modern mathematics is experiencing a complex stage in its development. In this connection, the book *Mathematics. The Loss of Certainty* [42], written by Moris Kline, Professor Emeritus of Mathematics of Courant Institute of Mathematical Sciences (New York University) is very symptomatic. Prof. Kline explains in his book the reasons of big difficulties in development of modern mathematics. The passion of mathematicians for abstractions and generalizations broke the contact of mathematics with natural sciences, which always were a source of mathematical ideas. Creating their abstract mathematical models, many "pure" mathematicians do not know practical applications of their models. Some "pure" mathematicians so went deep into their abstract models, which have a far relation to real physical world, that they forgotten the basic goal of mathematics to create models of real physical world. The "pure" mathematics turned into "the science for the science". This is one of reasons of the isolation of the "pure" mathematics from natural sciences! Of course, this remark does not concern to those branches of applied mathematics, in particular, discrete mathematics, which was created last decades in connection with development of natural sciences and technology.

But at the same time the interest in mathematics and new mathematical models in modern science has essentially increased. Because those models, which are created by the "pure" and "abstract" mathematicians, are frequently very far from real needs of modern natural sciences, the appeal of modern natural scientists to the sacral geometry is quite natural. Modern scientists find in the Sacral Geometry those simple and natural mathematical and geometrical models (regular polygons, platonic solids, the golden section, Fibonacci and Lucas numbers, Pascal triangle and binomial coefficients and so on), which underlie natural phenomena and many sciences.

The main idea of the harmony mathematics, which was created by the author and stated in the works [16–27,33–37], is to show that many old mathematical ideas (the golden section, Fibonacci numbers, Pascal triangle) that underlay the sacral geometry could become a source of useful and fruitful generalizations and applications. The harmony mathematics has a number of important consequences for modern science and mathematics. It can be considered as a new interdisciplinary direction of modern science that can influence on development of higher mathematics, natural sciences and mathematical education (Fig. 10).

First of all harmony mathematics fills a gap between natural sciences and higher mathematics, approaches higher mathematics to real problems of natural sciences and generates many fundamental concepts and theories that can be used effectively in natural sciences (physics, chemistry, biology, botany and so on). An influence of harmony mathematics on higher mathematics and natural sciences consists in development of such important scientific branch as *hyperbolic geometry*. It is clear that in parallel with the "hyperbolic world" that is based on the classical hyperbolic functions (Lobachevsky's hyperbolic geometry, Minkovsky's geometry, etc.), there is the "golden" hyperbolic world based on the hyperbolic Fibonacci and Lucas functions [18,22,24]. *The "golden" hyperbolic world exists objectively and independently on our consciousness and persistently shows itself in animate nature, in particular, in pine cones, heads of sunflower, pineapples, cactuses, baskets of various flowers in the form of Fibonacci and Lucas numbers that are found on surface of these biological objects (the phyllotaxis law).* A new class of hyperbolic functions, the hyperbolic Fibonacci and Lucas functions [18,22,24], can become a source for the development of the "golden" hyperbolic geometry, in particular, the "golden" Lobachevsky's and Minkovsky's geometry. In modern science the Ukrainian researcher Oleg Bodnar [41] approached to disclosing the laws of the "golden" hyperbolic world. It is necessary to be surprised only why in mathematics and theoretical physics during many centuries it was not given a proper attention to development of mathematical apparatus for modeling the "golden" hyperbolic world. Though, to honor of some physicists, in the end of 20th century the situation in the theoretical physics starts to change sharply. The articles [3–14] exhibit a substantial interest of modern theoretical physics and the physics of high energy particles in the golden section and "golden" physical hyperbolic world. The works of Mauldin and William, El Naschie, Vladimirov and other physicists show that it is impossible to imagine a future progress in physical and cosmological research without the golden section. The famous Russian physicist–theoretician Prof. Vladimirov (Moscow University) finished his book "Metaphysics" [11] with the following words:

"*Thus, it is possible to assert that in the theory of electroweak interactions there are relations that are approximately coincident with the "Golden Section" that play an important role in the various areas of science and art*"

$$
\begin{array}{ccccc}
 & & \textit{Natural Sciences} & & \\
 & & \Uparrow & & \\
\textit{Elementary Mathematics} & \Rightarrow & \textit{Harmony Mathematics} & \Rightarrow & \textit{Higher Mathematics} \\
 & & \Downarrow & & \\
 & & \textit{Mathematical Education} & &
\end{array}
$$

Fig. 10. Harmony Mathematics as a new interdisciplinary direction of modern science.

. On the other hand, harmony mathematics fills a gap between elementary mathematics and higher mathematics. That is why, as it is shown in [16–27,33–37], harmony mathematics can be used for the development of many fundamental concepts and theories of elementary and higher mathematics such as a concept of a number and number systems, a concept of measurement, a concept of elementary functions, a concept of matrix and so on. As is well known, elementary mathematics that was created in ancient period became a source of mathematical education. That is why one more important purpose of harmony mathematics is to approach mathematical education to real problems of natural sciences. The introduction of the golden section and harmony mathematics to educational curricula can increase an interest of students in study of mathematics because a process of mathematics study turns into the fascinating search of mathematical regularities of nature that surrounds us.

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