



On a new class of hyperbolic functions

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Abstract

This article presents the results of some new research on a new class of hyperbolic functions that unite the characteristics of the classical hyperbolic functions and the recurring Fibonacci and Lucas series. The hyperbolic Fibonacci and Lucas functions, which are the being extension of Binet's formulas for the Fibonacci and Lucas numbers in continuous domain, transform the Fibonacci numbers theory into "continuous" theory because every identity for the hyperbolic Fibonacci and Lucas functions has its discrete analogy in the framework of the Fibonacci and Lucas numbers. Taking into consideration a great role played by the hyperbolic functions in geometry and physics, ("Lobatchevski's hyperbolic geometry", "Four-dimensional Minkowski's world", etc.), it is possible to expect that the new theory of the hyperbolic functions will bring to new results and interpretations on mathematics, biology, physics, and cosmology. In particular, the result is vital for understanding the relation between transfiniteness i.e. fractal geometry and the hyperbolic symmetrical character of the disintegration of the neural vacuum, as pointed out by El Naschie [Chaos Solitons & Fractals 17 (2003) 631].

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1. Introduction. A role of the Golden Section in modern science

There is well-known fact that two mathematical constants of Nature, the π - and e -numbers, play a great role in mathematics and physics. These functions reflect some deep relations and regularities of the "Physical World" surrounding us. Their importance consists in the fact that they "generate" the main classes of so-called "elementary functions": *sin, cosine* (the π -number), *exponential, logarithmic and hyperbolic functions* (the e -number). It is impossible to imagine mathematics and physics without these functions. For example, there is the well-known greatest role of the classical hyperbolic functions in geometry (Hyperbolic Lobatchevski's geometry) and in cosmological researches (Four-dimensional Minkowski's world). However, there is the one more mathematical constant playing a great role in modeling of processes in living nature termed the *Golden Section, Golden Proportion, Golden Ratio, Golden Mean*. However, we should certify that a role of this mathematical constant is sometimes undeservedly humiliated in modern mathematics and mathematical education. What is a reason of this? Probably the reason consists in the wide usage of the Golden Section in so-called "esoteric sciences". There is the well-known fact that the basic symbols of esoteric (pentagram, pentagonal star, platonic solids etc.) are connected to the Golden Section closely. Moreover, the "materialistic" science together with its "materialistic" education had decided to "throw out" the Golden Section.

However, in modern science, an attitude towards the Golden Section and connected to its Fibonacci and Lucas numbers is changing very quickly. The outstanding discoveries of modern science (*Shechtman's quasi-crystals* [1], *Bodnar's theory of phyllotaxis* [2], *Soroko's law of structural harmony of systems* [3], *Butusov's resonance theory of the Solar system* [4], *Stakhov's algorithmic measurement theory* [5–7] and *Stakhov's codes of the Golden Proportion* [8]) based

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on the Golden Section have a revolutionary importance for development of modern science. These are enough convincing confirmation of the fact that human science approaches to uncovering one of the most complicated scientific notions, namely, the notion of *Harmony*, which according to Pythagoras, underlies the Universe.

In this connection, it is necessary to point out a great interest of modern physics in the Golden Section. The papers [9–16] present a special interest. In the paper [9] written by Maudlin and Willams in 1986 “*proved a theorem which at first sight may seem slightly paradoxical but we perceive as excitingly interesting. This theorem states that the Hausdorff dimension $d_c^{(0)}$ of a randomly constructed Cantor set is $d_c^{(0)} = \phi$, where $\phi = \frac{\sqrt{5}-1}{2}$ is the Golden Mean*” (quotation from [13]). El Naschie’s works [10–15] develop the Golden Mean applications into modern physics. In the paper [13] devoted to the role of the Golden Mean in quantum physics El Naschie conclude the following: “*In our opinion it is very worthwhile enterprise to follow the idea of Cantorian space-time with all its mathematical and physical ramifications. The final version may well be a synthesis between the results of quantum topology, quantum geometry and maybe also Rossler’s endorphysics, which like Nottale’s latest work makes extensive use of the ideas of Nelson’s stochastic mechanism*”.

In October 2003 in the Ukrainian town Vinnitsa, the International Conference “Problems of Harmony, Symmetry, and the Golden Section in Nature, Science, and Art” was held. The conference had interdisciplinary character and attracted attention of philosophers, mathematicians, physicists, economists, engineers, and linguists from Russia, Ukraine, Byelorussia, Canada, the USA and other countries. Participation of physicists-theoreticians became the significant event of the Conference. Three lectures of physicists-theoreticians were given at the planar session of the Conference and then these were published in the Proceedings of the Conference [16–18]. The paper [16] written by the famous Moscow physicist-theoretician Prof. Vladimirov (Department of Theoretical Physics, Moscow University) and his recent book [19] present the special interest because these researches are related to the quark theory that is based on the Icosahedron and the Golden Section. In the conclusion of his book, Prof. Vladimirov writes [19]: “*Thus, it is possible to assert, that in the theory of electroweak interactions there are the relations approximately coincident with the “Golden Section”, playing an important role in various areas of Science and Art*”.

Thus, in the Shechtman’s, Butusov’s, Mauldin and Williams’, El Naschie’s, Vladimirov’s works, the Golden Section occupied a firm place in modern physics and it is impossible to imagine the future progress in physical researches without the Golden Section.

2. Binet’s formulas and hyperbolic Fibonacci and Lucas functions

There are a number of fundamental results in the modern Fibonacci numbers theory. The one of them was found in 19th century by the famous French mathematician Binet. Studying the Golden Section, Fibonacci and Lucas numbers, he discovered two remarkable formulas, *Binet’s formulas*, connecting Fibonacci (F_n) and Lucas (L_n) numbers with the Golden Section $\alpha = \frac{1+\sqrt{5}}{2}$.

Let us consider so-called “*extended*” *Fibonacci and Lucas numbers* presented in the Table below:

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
F_{-n}	0	1	-1	2	-3	5	-8	13	-21	34	-55
L_n	2	1	3	4	7	11	18	29	47	76	123
L_{-n}	2	-1	3	-4	7	-11	18	-29	47	-76	123

As follows from the Table, the terms of the “*extended*” series F_n and L_n have a number of the wonderful mathematical properties. For example, for the odd $n = 2k + 1$ the terms of the sequences F_n and F_{-n} coincide, that is, $F_{2k+1} = F_{-2k-1}$, and even for the $n = 2k$ they are opposite by the sign, that is, $F_{2k} = -F_{-2k}$. This is same for the Lucas numbers L_n , here all is contrary, that is, $L_{2k} = L_{-2k}$; $L_{2k+1} = -L_{-2k-1}$.

Binet’s formulas give connections between the “*extended*” Fibonacci and Lucas numbers and the Golden Section that may be written in the following form:

$$F_n = \begin{cases} \frac{\alpha^{2k+1} + \alpha^{-(2k+1)}}{\sqrt{5}}, \\ \frac{\alpha^{2k} - \alpha^{-2k}}{\sqrt{5}}, \end{cases} \tag{1}$$

$$L_n = \begin{cases} \alpha^{2k} + \alpha^{-2k}, \\ \alpha^{2k+1} - \alpha^{-(2k+1)}, \end{cases} \tag{2}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ is the Golden Section and the discrete variable k takes its values from the set $0, \pm 1, \pm 2, \pm 3, \dots$

In [20], the discrete variable k , in the formulas (1) and (2), was replaced with the continuous variable x taken its values from the set of the real numbers and then the following continuous functions called *the hyperbolic Fibonacci and the Lucas functions* were introduced:

The hyperbolic Fibonacci sine

$$sF(x) = \frac{\alpha^{2x} - \alpha^{-2x}}{\sqrt{5}}. \quad (3)$$

The hyperbolic Fibonacci cosine

$$cF(x) = \frac{\alpha^{2x+1} + \alpha^{-(2x+1)}}{\sqrt{5}}. \quad (4)$$

The hyperbolic Lucas sine

$$sL(x) = \alpha^{2x+1} - \alpha^{-(2x+1)}. \quad (5)$$

The hyperbolic Lucas cosine

$$cL(x) = \alpha^{2x} + \alpha^{-2x}. \quad (6)$$

The correlation between the Fibonacci (F_n) and Lucas (L_n) numbers and the hyperbolic Fibonacci and the Lucas functions given by (3)–(6) consists in the following:

$$sF(k) = F_{2k}; \quad cF(k) = F_{2k+1}; \quad sL(k) = L_{2k+1}; \quad cL(k) = L_{2k}, \quad (7)$$

where $k = 0; \pm 1; \pm 2; \pm 3, \dots$

The book [21] is devoted to statement of the theory of the functions (3)–(6) and their applications in modern science and mathematics. The most important of them is a new (“continuous”) approach to the Fibonacci numbers theory based on the use of correlation (7), connecting the Fibonacci and Lucas numbers with the functions (3)–(6).

In the works [20] and [21], it was shown that, in contrast to the classical hyperbolic functions, the graph of the Fibonacci cosine (4) is asymmetrical concerning the axes of x , while the graph of the Lucas sine (5) is asymmetrical concerning the origin of coordinates. This confines the area of an effective application of a new class of hyperbolic functions given with (4)–(6).

The purpose of the present paper is to eliminate this deficiency of the functions (3)–(6) and to consider so-called *symmetrical representation* of the hyperbolic Fibonacci and Lucas functions. Together with the functions (3)–(6) introduced in [21], the symmetrical hyperbolic Fibonacci and Lucas functions introduced in the present paper may present a certain interest for modern theoretical physics taking into consideration a great role played by the Golden Section in modern physical researches.

3. Symmetrical representation of the hyperbolic Fibonacci and Lucas functions

Based on an analogy between Binet’s formulas (1) and (2) and the classical hyperbolic functions:

$$sh(x) = \frac{e^x - e^{-x}}{2}; \quad ch(x) = \frac{e^x + e^{-x}}{2},$$

we can give the following definitions of the hyperbolic Fibonacci and Lucas functions that are different from the definitions (3)–(6):

Symmetrical Fibonacci sine

$$sFs(x) = \frac{\alpha^x - \alpha^{-x}}{\sqrt{5}}. \quad (8)$$

Symmetrical Fibonacci cosine

$$cFs(x) = \frac{\alpha^x + \alpha^{-x}}{\sqrt{5}}. \quad (9)$$

Symmetrical Lucas sine

$$sLs(x) = \alpha^x - \alpha^{-x}. \tag{10}$$

Symmetrical Lucas cosine

$$cLs(x) = \alpha^x + \alpha^{-x}. \tag{11}$$

The Fibonacci and Lucas numbers are determined identically through the symmetrical Fibonacci and Lucas functions as the following:

$$F_n = \begin{cases} sFs(n), & \text{for } n = 2k; \\ cFs(n), & \text{for } n = 2k + 1; \end{cases} \quad L_n = \begin{cases} cLs(n), & \text{for } n = 2k. \\ sLs(n), & \text{for } n = 2k + 1. \end{cases}$$

The introduced above “symmetrical hyperbolic Fibonacci and Lucas functions” are connected with the classical hyperbolic functions by the following simple correlations:

$$sFs(x) = \frac{2}{\sqrt{5}}sh(\ln(\alpha) \cdot x); \quad cFs(x) = \frac{2}{\sqrt{5}}ch(\ln(\alpha) \cdot x);$$

$$sLs(x) = 2sh(\ln(\alpha) \cdot x); \quad cLs(x) = 2ch(\ln(\alpha) \cdot x).$$

It is easy to construct the graphs for the symmetrical Fibonacci and Lucas functions (Figs. 1 and 2). Their graphs have a symmetrical form and are similar to the graphs of the classical hyperbolic functions following from their definition of (8)–(11). Here is necessity to point out that, in the point $x = 0$, the symmetrical Fibonacci cosine $cFs(x)$ takes the value $cFs(0) = \frac{2}{\sqrt{5}}$, and the symmetrical Lucas cosine $cLs(x)$ takes the value $cLs(0) = 2$. It is also important to emphasize that the Fibonacci numbers F_n , with the even indexes ($n = 0, \pm 2, \pm 4, \pm 6, \dots$) are “inscribed” into the symmetrical Fibonacci sine $sFs(x)$ in the discrete points ($x = 0, \pm 2, \pm 4, \pm 6, \dots$) and Fibonacci numbers with the odd indexes ($n = \pm 1, \pm 3, \pm 5, \dots$) are “inscribed” into the symmetrical Fibonacci cosine $cFs(x)$ in the discrete points ($x = \pm 1, \pm 3, \pm 5, \dots$). In the other hand, the Lucas numbers with the even indexes are “inscribed” into the symmetrical Lucas cosine $cLs(x)$ in the discrete points ($x = 0, \pm 2, \pm 4, \pm 6, \dots$), and the Lucas numbers with the odd indexes are “inscribed” into the symmetrical Lucas cosine $sLs(x)$ in the discrete points ($x = \pm 1, \pm 3, \pm 5, \dots$). The difference between the symmetrical Fibonacci and Lucas functions in compare with the hyperbolic Fibonacci and Lucas functions (3)–(6) for which the connection with the Fibonacci and Lucas numbers is assigned by Eq. (7) consists in this emphasis.

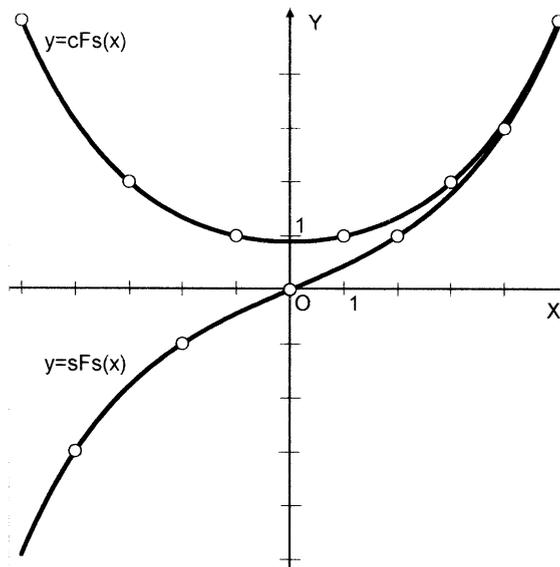


Fig. 1. Symmetrical hyperbolic Fibonacci sine and cosine.

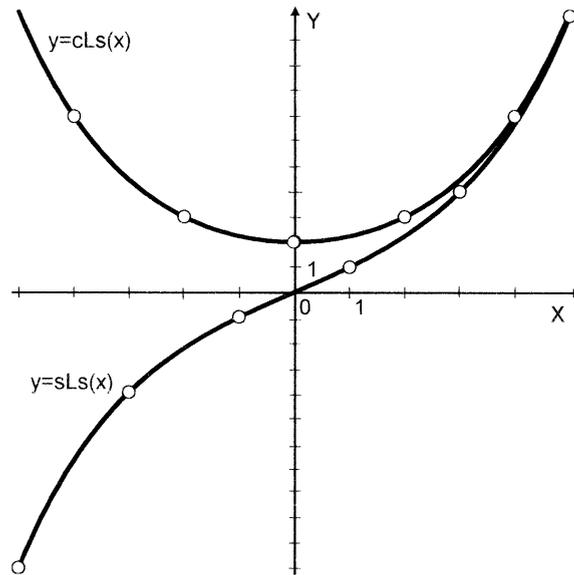


Fig. 2. Symmetrical hyperbolic Lucas sine and cosine.

The symmetrical hyperbolic Fibonacci and Lucas functions are connected among themselves by the following simple correlations:

$$sFs(x) = \frac{sLs(x)}{\sqrt{5}}; \quad cFs(x) = \frac{cLs(x)}{\sqrt{5}}.$$

4. Fibonacci’s property of the symmetrical hyperbolic Fibonacci and Lucas functions

Let us find some mathematical properties of the Fibonacci and Lucas functions introduced above.

Theorem 1. *The following correlations that are analogous to the recurrent equation for the Fibonacci numbers $F_{n+2} = F_{n+1} + F_n$ are valid for the symmetrical Fibonacci functions:*

$$sFs(x + 2) = cFs(x + 1) + sFs(x) \quad \text{and} \quad cFs(x + 2) = sFs(x + 1) + cFs(x).$$

Proof

$$\begin{aligned} cFs(x + 1) + sFs(x) &= \frac{\alpha^{x+1} + \alpha^{-x-1}}{\sqrt{5}} + \frac{\alpha^x - \alpha^{-x}}{\sqrt{5}} = \frac{\alpha^x(\alpha + 1) - \alpha^{-x}(1 - \alpha)}{\sqrt{5}} = \frac{\alpha^x\alpha^2 - \alpha^{-x}\alpha^{-2}}{\sqrt{5}} = \frac{\alpha^{x+2} - \alpha^{-x-2}}{\sqrt{5}} \\ &= sFs(x + 2); \end{aligned}$$

$$\begin{aligned} sFs(x + 1) + cFs(x) &= \frac{\alpha^{x+1} - \alpha^{-x-1}}{\sqrt{5}} + \frac{\alpha^x + \alpha^{-x}}{\sqrt{5}} = \frac{\alpha^x(\alpha + 1) + \alpha^{-x}(1 - \alpha)}{\sqrt{5}} = \frac{\alpha^x\alpha^2 + \alpha^{-x}\alpha^{-2}}{\sqrt{5}} = \frac{\alpha^{x+2} + \alpha^{-x-2}}{\sqrt{5}} \\ &= cFs(x + 2). \end{aligned}$$

Corollary from the Theorem 1. *Symmetrical exchanges $sFs(x)$ by $cFs(x)$ and $cFs(x)$ by $sFs(x)$ are possible.*

Theorem 2. *The following correlations that are analogous to the recurrent equation for the Fibonacci numbers $L_{n+2} = L_{n+1} + L_n$ are valid for the symmetrical Lucas functions:*

$$sLs(x + 2) = cLs(x + 1) + sLs(x) \quad \text{and} \quad cLs(x + 2) = sLs(x + 1) + cLs(x).$$

The proof is analogous to the Theorem 1.

Corollary from the Theorem 2. Symmetrical exchanges $sLs(x)$ by $cLs(x)$ and $cLs(x)$ by $sLs(x)$ are possible.

The rule of a symmetrical exchange is valid for all correlations connected symmetrical hyperbolic Fibonacci and Lucas functions. Therefore the authors have named a representation of the hyperbolic Fibonacci and Lucas functions, assigned by (8)–(11), as a symmetrical representation.

Theorem 3. The following correlations that are similar to the equation $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$ are valid for the symmetrical Fibonacci functions:

$$[sFs(x)]^2 - cFs(x + 1)cFs(x - 1) = -1 \quad \text{and} \quad [cFs(x)]^2 - sFs(x + 1) + sFs(x - 1) = 1.$$

Proof

$$\begin{aligned} [sFs(x)]^2 - sFs(x + 1)cFs(x - 1) &= \left[\frac{\alpha^x - \alpha^{-x}}{\sqrt{5}} \right]^2 - \frac{\alpha^{x+1} + \alpha^{-x-1}}{\sqrt{5}} \frac{\alpha^{x-1} + \alpha^{-x+1}}{\sqrt{5}} \\ &= \frac{\alpha^{2x} - 2 + \alpha^{-2x} - (\alpha^{2x} + \alpha^2 + \alpha^{-2} + \alpha^{-2x})}{5} = \frac{-(2 + \alpha + 1 + 2 - \alpha)}{5} = -1; \\ [cFs(x)]^2 - sFs(x + 1)sFs(x - 1) &= \left[\frac{\alpha^x - \alpha^{-x}}{\sqrt{5}} \right]^2 - \frac{\alpha^{x+1} - \alpha^{-x-1}}{\sqrt{5}} \frac{\alpha^{x-1} + \alpha^{-x+1}}{\sqrt{5}} \\ &= \frac{\alpha^{2x} + 2 + \alpha^{-2x} - (\alpha^{2x} - \alpha^2 - \alpha^{-2} + \alpha^{-2x})}{5} = \frac{(2 + \alpha + 1 + 2 - \alpha)}{5} = 1. \end{aligned}$$

Theorem 4. The following correlations that are similar to the equation $L_n^2 - 2(-1)^n = L_{2n}$ are valid for the symmetrical Lucas functions:

$$[cLs(x)]^2 + 2 = cLs(2x) \quad \text{and} \quad [cLs(x)]^2 - 2 = cLs(2x).$$

The proof is analogous to the Theorem 3.

Theorem 5. The following correlations that are similar to the equation $F_{n+1} + F_{n-1} = L_n$ are valid for the symmetrical Fibonacci and Lucas functions:

$$cFs(x + 1) + cFs(x - 1) = cLs(x) \quad \text{and} \quad sFs(x + 1) + sFs(x - 1) = sLs(x).$$

The proof is analogous to the Theorem 1.

Theorem 6. The following correlations that are similar to the equation $F_n + L_n = 2F_{n+1}$ are valid for the symmetrical Fibonacci and Lucas functions:

$$cFs(x) + sLs(x) = 2sFs(x + 1) \quad \text{and} \quad sFs(x) + cLs(x) = 2cFs(x + 1).$$

The proof is analogous to the Theorem 3.

Based on the definitions (6)–(9), it is easy to prove the following identities for the symmetrical Fibonacci and Lucas functions listed in Table 1.

5. Hyperbolic property of the symmetrical hyperbolic Fibonacci and Lucas functions

The symmetrical representation of the hyperbolic Fibonacci and Lucas functions has properties that are similar to the classical hyperbolic functions.

Theorem 7. The following correlations that are similar to the equation $[ch(x)]^2 - [sh(x)]^2 = 1$ are valid for the symmetrical Fibonacci functions:

$$[cFs(x)]^2 - [sFs(x)]^2 = 4/5.$$

Table 1

The identities for Fibonacci and Lucas numbers	The identities for the symmetrical hyperbolic Fibonacci and Lucas functions	
$F_{n+2} = F_{n+1} + F_n$	$sFs(x+2) = cFs(x+1) + sFs(x)$	$cFs(x+2) = sFs(x+1)cFs(x)$
$F_n = (-1)^n F_{-n}$	$sFs(x) = -sFs(-x)$	$cFs(x) = cFs(-x)$
$F_{n+3} + F_n = 2F_{n+2}$	$sFs(x+3) + cFs(x) = 2cFs(x+2)$	$cFs(x+3) + sFs(x) = 2sFs(x+2)$
$F_{n+3} - F_n = 2F_{n+1}$	$sFs(x+3) - cFs(x) = 2sFs(x+1)$	$cFs(x+3) - sFs(x) = 2cFs(x+1)$
$F_{n+6} - F_n = 4F_{n+3}$	$sFs(x+6) + sFs(x) = 4cFs(x+3)$	$cFs(x+6) + cFs(x) = 4sFs(x+3)$
$F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$	$[sFs(x)]^2 - cFs(x+1)cFs(x-1) = -1$	$[cFs(x)]^2 - sFs(x+1)sFs(x-1) = 1$
$F_{2n+1} = F_{n+1}^2 + F_n^2$	$cFs(2x+1) = [cFs(n+1)]^2 + [sFs(x)]^2$	$cFs(2x+1) = (sFs(n+1))^2 + [sFs(x)]^2$
$F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3$	$sFs(3x) = [cFs(x+1)]^3 + [sFs(x)]^3 - [cFs(x-1)]^3$	$cFs(3x) = [sFs(x+1)]^3 [cFs(x)]^3 - [sFs(x-1)]^3$
$L_{n+2} = L_{n+1} + L_n$	$sLs(x+2) = cLs(x+1) + sLs(x)$	$cLs(x+2) = sLs(x+1) + cLs(x)$
$L_n = (-1)^n L_{-n}$	$sLs(x) = -sLs(x)$	$cLs(x) = cLs(-x)$
$L_n^2 - 2(-1)^n = L_{2n}$	$[sLs(x)]^2 + 2 = cLs(2x)$	$[cLs(x)]^2 - 2 = cLs(2x)$
$L_n + L_{n+3} = 2L_{n+2}$	$sLs(x) + cLs(x+3) = 2sLs(x+2)$	$cLs(x) + sLs(x+3) = 2cLs(x+2)$
$L_{n+1}L_{n-1} - L_n^2 = -5(-1)^n$	$sLs(x+1)sLs(x-1) - [cLs(x)]^2 = -5$	$cLs(x+1)cLs(x-1) - [sLs(x)]^2 = 5$
$F_{n+3} - 2F_n = L_n$	$sFs(x+3) - 2cFs(x) = sLs(x)$	$cFs(x+3) - 2sFs(x) = cLs(x)$
$L_{n-1} + L_{n+1} = 5F_n$	$sLs(x-1) + sLs(x+1) = 5sFs(x)$	$cLs(x-1) + cLs(x+1) = 5cFs(x)$
$L_n + 5F_n = 2L_{n+1}$	$sLs(x) + 5cFs(x) = cLs(x+1)$	$cLs(x) + 5sFs(x) = sLs(x+1)$
$L_{n+1}^2 + L_n^2 = 5F_{2n+1}$	$[sLs(x+1)]^2 + [sLs(x)]^2 = 5cFs(x)$	$[cLs(x+1)]^2 + [cLs(x)]^2 = 5sFs(x)$

Proof

$$[cFs(x)]^2 - [sFs(x)]^2 = \left(\frac{\alpha^x + \alpha^{-x}}{\sqrt{5}}\right)^2 - \left(\frac{\alpha^x - \alpha^{-x}}{\sqrt{5}}\right)^2 = \frac{\alpha^{2x} + 2 + \alpha^{-2x} - \alpha^{2x} + 2 - \alpha^{-2x}}{\sqrt{5}} = \frac{4}{5}.$$

Theorem 8. The following correlations that are similar to the equation $ch(x+y) = ch(x)ch(y) + sh(x)sh(y)$ are valid for the symmetrical Fibonacci functions:

$$\frac{2}{\sqrt{5}}cFs(x+y) = cFs(x)cFs(y) + sFs(x)sFs(y).$$

Proof

$$\begin{aligned} cFs(x)cFs(y) + sFs(x)sFs(y) &= \frac{\alpha^x + \alpha^{-x}}{\sqrt{5}} \cdot \frac{\alpha^y + \alpha^{-y}}{\sqrt{5}} + \frac{\alpha^x - \alpha^{-x}}{\sqrt{5}} \cdot \frac{\alpha^y - \alpha^{-y}}{\sqrt{5}} \\ &= \frac{\alpha^{x+y} + \alpha^{x-y} + \alpha^{-x+y} + \alpha^{-x-y} + \alpha^{x+y} - \alpha^{x-y} - \alpha^{-x+y} + \alpha^{-x-y}}{5} = \frac{2(\alpha^{x+y} + \alpha^{-x-y})}{\sqrt{5}\sqrt{5}} \\ &= \frac{2}{\sqrt{5}}cFs(x+y). \end{aligned}$$

Theorem 9. The following correlations that are similar to the equation $ch(x-y) = ch(x)ch(y) - sh(x)sh(y)$ are valid for the symmetrical Fibonacci functions:

$$\frac{2}{\sqrt{5}}cFs(x-y) = cFs(x)cFs(y) - sFs(x)sFs(y).$$

Proof

$$\begin{aligned} cFs(x)cFs(y) - sFs(x)sFs(y) &= \frac{\alpha^x + \alpha^{-x}}{\sqrt{5}} \cdot \frac{\alpha^y + \alpha^{-y}}{\sqrt{5}} - \frac{\alpha^x - \alpha^{-x}}{\sqrt{5}} \cdot \frac{\alpha^y - \alpha^{-y}}{\sqrt{5}} \\ &= \frac{\alpha^{x+y} + \alpha^{x-y} + \alpha^{-x+y} + \alpha^{-x-y} - \alpha^{x+y} + \alpha^{x-y} + \alpha^{-x+y} - \alpha^{-x-y}}{5} = \frac{2(\alpha^{x+y} + \alpha^{-x-y})}{\sqrt{5}\sqrt{5}} \\ &= \frac{2}{\sqrt{5}}cFs(x-y). \end{aligned}$$

Theorem 10. *The following correlations are valid for the derivative hyperbolic Fibonacci functions:*

$$[ch(x)]^{(n)} = \begin{cases} sh(x), & \text{for } n = 2k + 1 \\ ch(x), & \text{for } n = 2k \end{cases}; \quad [sh(x)]^{(n)} = \begin{cases} ch(x), & \text{for } n = 2k + 1 \\ sh(x), & \text{for } n = 2k \end{cases}.$$

Proof

$$\begin{aligned} [cFs(x)]' &= \left(\frac{\alpha^x + \alpha^{-x}}{\sqrt{5}}\right)' = \frac{(\alpha^x + \alpha^{-x})'}{\sqrt{5}} = \frac{\alpha^x \ln(\alpha) - \alpha^{-x} \ln(\alpha)}{\sqrt{5}} = \ln(\alpha) sFs(x) \\ [sFs(x)]' &= \left(\frac{\alpha^x - \alpha^{-x}}{\sqrt{5}}\right)' = \frac{(\alpha^x - \alpha^{-x})'}{\sqrt{5}} = \frac{\alpha^x \ln(\alpha) + \alpha^{-x} \ln(\alpha)}{\sqrt{5}} = \ln(\alpha) cFs(x) \\ [cFs(x)]'' &= (\ln(\alpha) sFs(x))' = \ln(\alpha) \left(\frac{\alpha^x - \alpha^{-x}}{\sqrt{5}}\right)' = \ln(\alpha) \frac{\alpha^x \ln(\alpha) + \alpha^{-x} \ln(\alpha)}{\sqrt{5}} = [\ln(\alpha)]^2 cFs(x) \\ [sFs(x)]'' &= (\ln(\alpha) cFs(x))' = \ln(\alpha) \left(\frac{\alpha^x + \alpha^{-x}}{\sqrt{5}}\right)' = \ln(\alpha) \frac{\alpha^x \ln(\alpha) - \alpha^{-x} \ln(\alpha)}{\sqrt{5}} = [\ln(\alpha)]^2 sFs(x) \\ &\dots\dots\dots \\ [cFs(x)]^{(n)} &= \begin{cases} (\ln(\alpha))^n sFs(x), & \text{for } n = 2k + 1 \\ (\ln(\alpha))^n cFs(x), & \text{for } n = 2k \end{cases} \\ [sFs(x)]^{(n)} &= \begin{cases} (\ln(\alpha))^n cFs(x), & \text{for } n = 2k + 1 \\ (\ln(\alpha))^n sFs(x), & \text{for } n = 2k \end{cases} \end{aligned}$$

In Table 2, some known properties of the classical hyperbolic functions and the appropriate properties of the Symmetrical Fibonacci and Lucas functions that are indicated for comparison. \square

Theorem 11. *The following equations that are similar to Moivre’s equation are valid for the symmetrical Fibonacci and Lucas functions:*

$$\begin{aligned} [cFs(x) \pm sFs(x)]^n &= \left(\frac{2}{\sqrt{5}}\right)^{n-1} [cFs(nx) \pm sFs(nx)]; \\ [cLs(x) \pm sLs(x)]^n &= 2^{n-1} [cFs(nx) \pm sFs(nx)]. \end{aligned}$$

The proof is analogous to Theorem 7.

6. Bodnar’s geometry

In botanic, there is well-known *phyllotaxis* phenomenon displayed in many dense-packed botanical objects such as head of sunflower, pinecone, pineapple, and cactus etc. To characterize such phyllotaxis objects the number ratios of the left and right spirals that are observed on the surface of the phyllotaxis objects is usually used. These ratios are equal to the ratios of the adjacent Fibonacci numbers, that is:

$$\frac{F_{n+1}}{F_n} : \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \dots \rightarrow \alpha = \frac{1 + \sqrt{5}}{2}. \tag{12}$$

For example, the head of sunflower may have phyllotaxis orders given by the Fibonacci’s ratios $\frac{89}{55}$, $\frac{144}{89}$ and even $\frac{233}{144}$.

Observing phyllotaxis object in the completed condition and enjoying by the well organized picture on its surface, we always put a question: how is Fibonacci lattice formed on its surface during its growth. This problem presents by itself one of the most intriguing puzzles of phyllotaxis. Its essence consists in the fact that a majority of bio-form kinds change their phyllotaxis orders during their growth. It is known, for example, that sunflower disks located on different levels of the same stalk have different phyllotaxis orders. Moreover, the older a disk is, the higher its phyllotaxis order. It means that a natural modification (an increase) of symmetry is occurred during phyllotaxis and this modification of symmetry is carried out under the law:

$$\frac{2}{1} \rightarrow \frac{3}{2} \rightarrow \frac{5}{3} \rightarrow \frac{8}{5} \rightarrow \frac{13}{8} \rightarrow \frac{21}{13} \rightarrow \dots \tag{13}$$

Table 2

Classical hyperbolic functions	Symmetrical Fibonacci functions	Symmetrical Lucas functions
$[ch(x)]^2 - [sh(x)]^2 = 1$ $ch(x \pm y) = ch(x)ch(y) \pm sh(x)sh(y)$ $sh(x \pm y) = sh(x)ch(y) \pm ch(x)sh(y)$ $ch(2x) = [ch(x)]^2 + [sh(x)]^2$ $sh(2x) = 2sh(x) \cdot ch(x)$	$[cFs(x)]^2 - [sFs(x)]^2 = 4/5$ $\frac{2}{\sqrt{5}}cFs(x \pm y) = cFs(x)cFs(y) \pm sFs(x)sFs(y)$ $\frac{2}{\sqrt{5}}sFs(x \pm y) = sFs(x)cFs(y) \pm cFs(x)sFs(y)$ $\frac{2}{\sqrt{5}}cFs(2x) = [cFs(x)]^2 + [sFs(x)]^2$ $\frac{2}{\sqrt{5}}sFs(2x) = sFs(x) \cdot cFs(x)$	$[cLs(x)]^2 - [sLs(x)]^2 = 4$ $2cLs(x \pm y) = cLs(x)cLs(y) \pm sLs(x)sLs(y)$ $2sLs(x \pm y) = sLs(x)cLs(y) \pm cLs(x)sLs(y)$ $2cLs(2x) = [cLs(x)]^2 + [sLs(x)]^2$ $sLs(2x) = sLs(x) \cdot cLs(x)$
$[ch(x)]^{(n)} = \begin{cases} sh(x), & \text{for } n = 2k + 1 \\ ch(x), & \text{for } n = 2k \end{cases}$ $[sh(x)]^{(n)} = \begin{cases} ch(x), & \text{for } n = 2k + 1 \\ sh(x), & \text{for } n = 2k \end{cases}$	$[cFs(x)]^{(n)} = \begin{cases} (ln(\alpha))^n sFs(x), & \text{for } n = 2k + 1 \\ (ln(\alpha))^n cFs(x), & \text{for } n = 2k \end{cases}$ $[sFs(x)]^{(n)} = \begin{cases} (ln(\alpha))^n cFs(x), & \text{for } n = 2k + 1 \\ (ln(\alpha))^n sFs(x), & \text{for } n = 2k \end{cases}$	$[cLs(x)]^{(n)} = \begin{cases} (ln(\alpha))^n sLs(x), & \text{for } n = 2k + 1 \\ (ln(\alpha))^n cLs(x), & \text{for } n = 2k \end{cases}$ $[sLs(x)]^{(n)} = \begin{cases} (ln(\alpha))^n cLs(x), & \text{for } n = 2k + 1 \\ (ln(\alpha))^n sLs(x), & \text{for } n = 2k \end{cases}$
$\int \int \int_n ch(x) dx = \begin{cases} sh(x), & \text{for } n = 2k + 1 \\ ch(x), & \text{for } n = 2k \end{cases}$ $\int \int \int_n sh(x) dx = \begin{cases} ch(x), & \text{for } n = 2k + 1 \\ sh(x), & \text{for } n = 2k \end{cases}$	$\int \int \int_n cFs(x) dx = \begin{cases} (ln(\alpha))^{-n} sFs(x), & \text{for } n = 2k + 1 \\ (ln(\alpha))^{-n} cFs(x), & \text{for } n = 2k \end{cases}$ $\int \int \int_n sFs(x) dx = \begin{cases} (ln(\alpha))^{-n} cFs(x), & \text{for } n = 2k + 1 \\ (ln(\alpha))^{-n} sFs(x), & \text{for } n = 2k \end{cases}$	$\int \int \int_n cLs(x) dx = \begin{cases} (ln(\alpha))^{-n} sLs(x), & \text{for } n = 2k + 1 \\ (ln(\alpha))^{-n} cLs(x), & \text{for } n = 2k \end{cases}$ $\int \int \int_n sLs(x) dx = \begin{cases} (ln(\alpha))^{-n} cLs(x), & \text{for } n = 2k + 1 \\ (ln(\alpha))^{-n} sLs(x), & \text{for } n = 2k \end{cases}$

The modification of the orders of phyllotaxis according to (13) is named as *dynamic symmetry*. All above-stated data also make an essence of the well known “*puzzle of phyllotaxis*”. Researching this problem, many scientists assume that the phenomenon of the dynamical symmetry of the phyllotaxis objects has fundamental interdisciplinary importance. According to Vernadski’s opinion, the problem of a biological symmetry is the *key problem of biology*.

Thus, the phenomenon of the dynamic symmetry (13) finds out a special role in the geometric problem of phyllotaxis. Anyone may assume that the numerical regularity (13) reflects some general geometric laws, which, probably, make an essence of a secret of the dynamic mechanism of phyllotaxis and their uncovering would have a great importance for solution of the phyllotaxis phenomenon in the whole.

The Ukrainian architect Oleg Bodnar, in his brilliant book [2], recently solved the “*puzzle of phyllotaxis*” given by (13). Modelling the growing of the phyllotaxis objects, he used a notion of the “*hyperbolic turn*” used in the hyperbolic geometry and so-called “golden” hyperbolic functions, which differ from the introduced above symmetrical hyperbolic Fibonacci functions only by constant coefficients.

7. Conclusion

Thus, the main result of the present paper, the paper [20] and the book [21] is a strong mathematical proof of the deep connection between the Golden Section, Fibonacci and Lucas numbers, and hyperbolic functions. A new class of the hyperbolic functions based on the Golden Section could have far going consequences for future progress of mathematics, physics, biology and cosmology. In the first place, the hyperbolic Fibonacci and Lucas functions which are the being extension of Binet’s formulas for the Fibonacci and Lucas numbers in continuous domain transform the Fibonacci numbers theory into “continuous” theory because every identity for the hyperbolic Fibonacci and Lucas functions has its discrete analogy in the framework of the Fibonacci and Lucas number theory. In the other words, the theory of Fibonacci and Lucas numbers are being the “discrete” case of the theory of the hyperbolic Fibonacci and Lucas functions. Considering the fundamental role of the classical hyperbolic functions in the mathematical tools of the modern science, it is possible to suppose that the new theory of the hyperbolic functions will bring the new results and interpretations in various spheres of natural science.

In 1996, the famous mathematical journal “The Fibonacci Quarterly” published the paper “On Fibonacci hyperbolic trigonometry and modified numerical triangles” written by the well-known Fibonacci-mathematician Zdzislaw W. Trzaska [22]. Anyone may welcome the interest of the Fibonacci-mathematicians in the hyperbolic Fibonacci functions, however. . . The comparison of Trzaska’s paper [22] and Stakhov and Tkachenko’s paper [20] showed that in the part concerning the hyperbolic Fibonacci functions, Mr. Trzaska’s used on 100% the material of Stakhov and Tkachenko’s paper [20] but Trzaska’s paper [22] does no have a reference to the paper [20] that creates an illusion of Trzaska’s priority over the hyperbolic Fibonacci functions.

As the paper [20] was published in 1993, that is, in 3 years before publication of Trzaska’s paper [22], as it follows from this comparison that Stakhov and Tkachenko has priority over the hyperbolic Fibonacci and Lucas functions is indisputable and Trzaska’s paper [22] is simple misunderstanding.

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