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# Fibonacci matrices, a generalization of the "Cassini formula", and a new coding theory

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**Abstract**

We consider a new class of square Fibonacci $(p+1)\times(p+1)$-matrices, which are based on the Fibonacci $p$-numbers $(p = 0, 1, 2, 3, \ldots)$, with a determinant equal to $+1$ or $-1$. This unique property leads to a generalization of the "Cassini formula" for Fibonacci numbers. An original Fibonacci coding/decoding method follows from the Fibonacci matrices. In contrast to classical redundant codes a basic peculiarity of the method is that it allows to correct matrix elements that can be theoretically unlimited integers. For the simplest case the correct ability of the method is equal 93.33% which exceeds essentially all well-known correcting codes.


## 1. Introduction

In the last decades the theory of Fibonacci numbers [1,8] was complemented by the theory of the so-called *Fibonacci Q-matrix* [1,2]. The latter is a square $2\times 2$ matrix of the following form:

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \quad (1)$$

In [1] the following property of the $n$th power of the $Q$-matrix was proved

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \quad (2)$$

$$\text{Det } Q^n = F_{n+1}F_{n-1} - F_n^2 = (-1)^n, \quad (3)$$

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots, F_{n-1}, F_n, F_{n+1}$ are Fibonacci numbers given with the following recurrence relation:

$$F_{n+1} = F_n + F_{n-1} \quad (4)$$

with the initial terms

$$F_1 = F_2 = 1. \quad (5)$$

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Note that identity (4) is called "*Cassini formula*" in honor of the well-known 17th century astronomer Giovanni Cassini (1625–1712) who derived this formula.

In 1977 the author introduced so-called *Fibonacci p-numbers* [3]. For a given integer $p = 0, 1, 2, 3, \ldots$ the Fibonacci $p$-numbers are given with the following recurrence relation:

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \quad \text{with } n > p+1 \tag{6}$$

with the initial terms

$$F_p(1) = F_p(2) = \cdots = F_p(p) = F_p(p+1) = 1. \tag{7}$$

In [4] the notion of the $Q_p$-matrices ($p = 0, 1, 2, 3, \ldots$) was introduced. This notion is a generalization of the $Q$-matrix (1) and is connected to the Fibonacci $p$-numbers (6) and (7).

The main purpose of the present article is to develop a theory of the $Q_p$-matrices. The next purpose is to give a generalization of the "Cassini formula" (4) that follows from the theory of the $Q_p$-matrices. Also a new approach to a coding theory, which is based on the $Q_p$-matrices, is considered.

## 2. Some properties of the Fibonacci *p*-numbers

It is clear that the recurrence formula (6) with the initial terms (7) "generates" an infinite number of recurrent sequences. In particular, for the case $p = 0$ recurrence relation (6) and (7) reduces to the following:

$$F_0(n) = F_0(n-1) + F_0(n-1) \quad \text{with } n > 1, \tag{8}$$

$$F_0(1) = 1. \tag{9}$$

This recurrence relation "generates" the binary numbers: $1, 2, 4, 8, \ldots, 2^{n-1}, \ldots$.

For the case $p = 1$ recurrence relation (6) and (7) reduces to the following:

$$F_1(n) = F_1(n-1) + F_1(n-2) \quad \text{with } n > 2, \tag{10}$$

$$F_1(1) = F_1(2) = 1. \tag{11}$$

This recurrent relation "generates" the classical Fibonacci numbers $F_1(n) = F_n$

$$1, 1, 2, 3, 5, 8, 13, \ldots \tag{12}$$

It is clear that for general case the recurrence relation (6) and (7) "generates" infinite number of numerical series, which are a wide generalization of the classical Fibonacci numbers.

Like to the classical Fibonacci numbers (12) the Fibonacci $p$-numbers for the case $p > 0$ allow their extension to the negative values of the argument $n$. For calculation of the Fibonacci $p$-numbers $F_p(0), F_p(-1), F_p(-2), \ldots, F_p(-p), \ldots, F_p(-2p+1)$ we will use recurrence relation (6) and initial terms (7). Representing the Fibonacci $p$-numbers $F_p(p+1)$ in the form (6) we get

$$F_p(p+1) = F_p(p) + F_p(0). \tag{13}$$

As according to (7) $F_p(p) = F_p(p+1) = 1$ it follows from (13) that $F_p(0) = 0$.

Continuing this process, that is, representing the Fibonacci $p$-numbers $F_p(p), F_p(p-1), \ldots, F_p(2)$ in the form (6) we get

$$F_p(0) = F_p(-1) = F_p(-2) = \cdots = F_p(-p+1) = 0. \tag{14}$$

Let us represent now the number $F_p(1)$ in the form:

$$F_p(1) = F_p(0) + F_p(-p). \tag{15}$$

As $F_p(1) = 0$ and $F_p(0) = 0$ it follows from (15) that

$$F_p(-p) = 1. \tag{16}$$

Representing the Fibonacci $p$-numbers $F_p(0), F_p(-1), \ldots, F_p(-p+1)$ in the form (6) we can find

$$F_p(-p-1) = F_p(-p-2) = \cdots = F_p(-2p+1) = 0. \tag{17}$$

Continuing this process we can get all values of the Fibonacci $p$-numbers $F_p(n)$ for the negative values of $n$ (see Table 1).

Table 1
Fibonacci $p$-numbers

| $n$ | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | −1 | −2 | −3 | −4 | −5 | −6 | −7 | −8 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_1(n)$ | 13 | 8 | 5 | 3 | 2 | 1 | 1 | 0 | 1 | −1 | 2 | −3 | 5 | −8 | 13 | −21 |
| $F_2(n)$ | 6 | 4 | 3 | 2 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | −1 | 1 | 1 | −2 | 0 |
| $F_3(n)$ | 4 | 3 | 2 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | −1 | 1 | 0 |
| $F_4(n)$ | 3 | 2 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | −1 |
| $F_5(n)$ | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |

The following property for the Fibonacci $p$-numbers is proved in [3]:

$$F_p(1) + F_p(2) + F_p(3) + \cdots + F_p(n) = F_p(n + p + 1) - 1. \tag{18}$$

Note that the formula (18) includes a number of the remarkable formulas of discrete mathematics. For example, for the case $p = 0$, this formula reduces to the following well-known formula for the binary numbers:

$$2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 1.$$

As it is mentioned above, for the case $p = 1$ the Fibonacci $p$-numbers coincide with the classical Fibonacci numbers $F_n$. Therefore, identity (18) reduces to the following formula:

$$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$$

that is well known from the Fibonacci numbers theory [1].

## 3. Some properties of the $Q$-matrix

Let us represent matrix (2) in the following form:

$$Q^n = \begin{pmatrix} F_n + F_{n-1} & F_{n-1} + F_{n-2} \\ F_{n-1} + F_{n-2} & F_{n-2} + F_{n-3} \end{pmatrix} = \begin{pmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{pmatrix} + \begin{pmatrix} F_{n-1} & F_{n-2} \\ F_{n-2} & F_{n-3} \end{pmatrix} \tag{19}$$

or

$$Q^n = Q^{n-1} + Q^{n-2}. \tag{20}$$

Let us write the expression (20) in the form:

$$Q^{n-2} = Q^n - Q^{n-1}. \tag{21}$$

It is proved in [4] the following property of the $Q$-matrix:

$$Q^n Q^m = Q^m Q^n = Q^{n+m}. \tag{22}$$

The explicit forms of the matrices $Q^n (n = 0, \pm 1, \pm 2, \pm 3, \ldots)$ obtained by means of recurrence relations (20) and (21) are given in Table 2.

Note that Table 2 gives the "direct" matrices $Q^n$ and their "inverse" matrices $Q^{-n}$ in explicit form. Comparing the "direct" and "inverse" Fibonacci matrices $Q^n$ and $Q^{-n}$ given in Table 2 it is easy to see that there is a very simple method to get the "inverse" matrix $Q^{-n}$ from its "direct" matrix $Q^n$. In fact, if the power of the "direct" matrix $Q^n$ given with (2) is odd number ($n = 2k$) then for getting of its "inverse" matrix $Q^{-n}$ it is necessary to rearrange in matrix

Table 2
The explicit forms of the matrices $Q^n$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $Q^n$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ | $\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$ | $\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$ | $\begin{pmatrix} 8 & 5 \\ 5 & 3 \end{pmatrix}$ | $\begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}$ | $\begin{pmatrix} 21 & 13 \\ 13 & 8 \end{pmatrix}$ |
| $Q^{-n}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ | $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ | $\begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$ | $\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$ | $\begin{pmatrix} -3 & 5 \\ 5 & -8 \end{pmatrix}$ | $\begin{pmatrix} 5 & -8 \\ -8 & 13 \end{pmatrix}$ | $\begin{pmatrix} -8 & 13 \\ 13 & -21 \end{pmatrix}$ |

Table 3

| $n$ | 6 | **5** | **4** | 3 | 2 | **1** | **0** | −1 | −2 | −3 | **−4** | **−5** | **−6** | **−7** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_{n+1}$ | 13 | **8** | **5** | 3 | 2 | **1** | **1** | 0 | 1 | −1 | 2 | **−3** | **5** | **−8** |
| $F_n$ | **8** | **5** | **3** | 2 | 1 | **1** | **0** | 1 | −1 | 2 | −3 | **5** | **−8** | **13** |

(2) its diagonal elements $F_{n+1}$ and $F_{n-1}$ and to take its diagonal elements $F_n$ with an opposite sign. It means that for the case $n = 2k$ the "inverse" matrix $Q^{-n}$ has the following form:

$$Q^{-2k} = \begin{pmatrix} F_{2k-1} & -F_{2k} \\ -F_{2k} & F_{2k+1} \end{pmatrix}. \tag{23}$$

For the case $n = 2k + 1$ for getting the "inverse" matrix $Q^{-n}$ from the "direct" matrix $Q^n$ it is necessary to rearrange in (2) the diagonal elements $F_{n+1}$ and $F_{n-1}$ and to take them with an opposite sign, that is

$$Q^{-(2k+1)} = \begin{pmatrix} -F_{2k} & F_{2k+1} \\ F_{2k+1} & -F_{2k+2} \end{pmatrix}. \tag{24}$$

Other method to get the matrix $Q^{-n}$ follows directly from expression (2). For that it is necessary to present two sequences of Fibonacci numbers shifted one to another in one column (Table 3).

If we select number $n = 1$ in the first row of Table 3 and then four Fibonacci numbers in two next rows (under numbers 1 and 0 of the first rows) we can see that a totality of four Fibonacci numbers forms the $Q$-matrix. Moving through Table 3 to the left regarding to the $Q$-matrix we will get consecutively the matrices $Q^2, Q^3, \ldots, Q^n$. Moving to the right regarding to the $Q$-matrix we will get consecutively the matrices $Q^0, Q^{-1}, \ldots, Q^{-n}$. As example we can see in Table 3 the matrix $Q^5$ and its "inverse" matrix $Q^{-5}$, which are singled out with fatty print.

## 4. The generalized Fibonacci matrices

In [4] it was introduced the generalized Fibonacci matrices designed as $Q_p$-matrices. For a given $p(p = 0, 1, 2, 3, \ldots)$ let us introduce the following definition for the $Q$-matrix:

$$Q_p = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{25}$$

Note that the $Q_p$-matrix is a square $(p + 1) \times (p + 1)$ matrix. It contains a $p \times p$ identity matrix bordered by the last row of 0's and the first column, which consists of 0's bordered by 1's. For $p = 0, 1, 2, 3, 4$ the $Q_p$-matrices have the following forms, respectively:

$$Q_0 = (1); \quad Q_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = Q, \quad Q_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$
$$Q_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad Q_4 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us compare the neighboring matrices $Q_4$ and $Q_3$. It is easy to see that the matrix $Q_4$ reduces to the matrix $Q_3$ if we cross out in the matrix $Q_4$ the last (5th) column and the next to last ($4^s$) row. Note that we have 1 on the intersection of the 5th column and the $4^s$ row. Because the sum $5+4$ is equal to the odd number 9 it means that determinant of the matrix $Q_3$ differs from the determinant of the matrix of $Q_3$ only by a sign [5], that is

$$\text{Det } Q_4 = -\text{Det } Q_3.$$

By analogy it is easy to prove the following correlations for determinants of the neighboring Fibonacci $Q_p$-matrices

$$\text{Det } Q_3 = -\text{Det } Q_2; \quad \text{Det}\, Q_2 = -\text{Det}\, Q_1.$$

Taking in consideration that $\text{Det}\, Q_0 = 1$ and $\text{Det}\, Q_1 - 1$ we can get the following unique mathematical property of the $Q_p$-matrixes (25) in general case.

**Theorem 1.** *For a given integer $p = 0, 1, 2, 3, \ldots$ we have*

$$Det\, Q_p = (-1)^p. \tag{26}$$

*Let us consider now the nth power of the $Q_p$-matrix, $Q_p^n$. In [4] it was proved the following theorem.*

**Theorem 2.** *For a given integer $p = 0, 1, 2, 3, \ldots$ we have*

$$Q_p^n = \begin{pmatrix} F_p(n+1) & F_p(n) & \cdots & F_p(n-p+2) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-p) & \cdots & F_p(n-2p+2) & F_p(n-2p+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_p(n-1) & F_p(n-2) & \cdots & F_p(n-p) & F_p(n-p-1) \\ F_p(n) & F_p(n-1) & \cdots & F_p(n-p+1) & F_p(n-p) \end{pmatrix}, \tag{27}$$

*where $F_p(n)$ is Fibonacci p-number, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$*

*It was proved in [4] the following theorems for the matrix $Q_p^n$.*

**Theorem 3.** *For a given integer $p = 0, 1, 2, 3, \ldots$ we have*

$$Q_p^n \times Q_p^m = Q_p^m \times Q_p^n = Q_p^{m+n}. \tag{28}$$

**Theorem 4.** *For a given integer $p = 0, 1, 2, 3, \ldots$ we have*

$$Q_p^n = Q_p^{n-1} + Q_p^{n-p-1}. \tag{29}$$

## 5. A generalization of the "Cassini formula"

Let us consider the matrix $Q_p^n$ given with (27). It follows from matrix theory [5] the following property for the determinant of the matrix $Q_p^n$:

$$\text{Det}\, Q_p^n = (\text{Det}\, Q_p)^n. \tag{30}$$

Then using (26) it is easy to prove the following theorem.

**Theorem 5.** *For a given integer $p = 0, 1, 2, 3, \ldots$ we have*

$$Det\, Q_p^n = (-1)^{pn}, \tag{31}$$

*where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$*

Let us compare formula (29) to formula (4). Because identity (4) expresses the "Cassini formula" for Fibonacci numbers then we can conclude that identity (29) expresses the generalized "*Cassini formula*". Note that Theorems 2 and 3 give unlimited opportunities for "Fibonacci enthusiasts" because they allow to get an infinite number of fundamental identities similar to (4), which connects the Fibonacci $p$-numbers $F_p(n)$ between themselves.

Let us consider the "Cassini formula" for the Fibonacci 2-numbers ($p = 2$) that are given with the following recurrence relation:

$$F_2(n) = F_2(n-1) + F_2(n-3) \quad \text{with } n > 2 \tag{32}$$

with the initial terms

$$F_2(1) = F_2(2) = F_3(2) = 1. \tag{33}$$

The matrix $Q_2^n$ for the case $p = 2$ looks as the following:

$$Q_2^n = \begin{pmatrix} F_2(n+1) & F_2(n) & F_2(n-1) \\ F_2(n-1) & F_2(n-2) & F_2(n-3) \\ F_2(n) & F_2(n-1) & F_2(n-2) \end{pmatrix}, \tag{34}$$

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

We can calculate a determinant of the matrix (34) using rules of matrix algebra [5] and then using (29) we can write the following identity for the determinant of matrix (34):

$$\begin{aligned} \operatorname{Det} Q_2^n = {} & F_2(n+1)[F_2(n-2)F_2(n-2) - F_2(n-1)F_2(n-3)] + F_2(n)[F_2(n)F_2(n-3) \\ & - F_2(n-1)F_2(n-2)] + F_2(n-1)[F_2(n-1)F_2(n-1) - F_2(n)F_2(n-2)] = 1, \end{aligned} \tag{35}$$

where $F_2(n-3)$, $F_2(n-2)$, $F_2(n-1)$, $F_2(n)$, $F_2(n+1)$ are five adjacent Fibonacci 2-numbers given by recurrence relation (32) and (33).

Continuing this process we can calculate determinants of the matrixes $Q_p^n$ for the cases $p = 3, 4, 5, \ldots$. Using (29) we can write the "Cassini formulas" for the cases $p = 3, 4, 5, \ldots$

Now it is difficult to predict applications of the $Q_p$-matrices given with (25) and (27) in physics but it is clear that the $Q_p$-matrices with a unique property (29) are of general interest for matrix theory [5] and especially for the "Fibonacci numbers theory" [1] because they extend the area of "Fibonacci's research".

## 6. A new coding theory

### 6.1. Fibonacci coding/decoding method

The introduced in [4] Fibonacci $Q_p$-matrices allow to develop the following application to coding theory [6].

Let us represent an initial message in the form of the square matrix $M$ of the size $(p+1) \times (p+1)$ where $p = 0, 1, 2, 3, \ldots$ Let us choice the Fibonacci $Q_p$-matrix. $Q_p^n$ of the size $(p+1) \times (p+1)$ as a *coding matrix* and its inverse matrix $Q_p^{-n}$ of the same size as a *decoding matrix*.

Let us consider now the following transformations based on matrix multiplication (see Table 4).

We will name a transformation $M \times Q_p^n = E$ as *Fibonacci coding* and a transformation $E \times Q_p^{-n} = M$ asy *Fibonacci decoding*. We will name the matrix $E$ as *code matrix*.

The coding/decoding method given by Table 4 ensures infinite variants of possible transformation of the initial matrix $M$ as every Fibonacci coding matrix $Q_p^n$ and its inverse matrix $Q_p^{-n} (p = 1, 2, 3, \ldots, n = 1, 2, 3, \ldots)$ give their own Fibonacci coding/decoding method according to Table 4. Note that for the case $p = 0$ the matrix $Q_p$ reduces to the trivial matrix $Q_0 = (1)$ and for this case the coding/decoding method given by Table 4 is trivial. For the case $p = 1$ the matrix $Q_p$ reduces to the classical $Q$-matrix (1) [1].

### 6.2. Example of the Fibonacci coding/decoding method

Let us consider now the simplest Fibonacci coding/decoding method based on application of the classical Fibonacci $Q$-matrix given by (1) and (2). Let us represent the initial message $M$ in the form of the following $2 \times 2$ matrix:

$$M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}. \tag{36}$$

Table 4
Fibonacci coding/decoding method

| Coding | Decoding |
|---|---|
| $M \times Q_P^n = E$ | $E \times Q_P^{-n} = M$ |

Let us assume that all elements of the matrix (36) are positive integers, that is

$$m_1 > 0; \quad m_2 > 0; \quad m_3 > 0; \quad m_4 > 0. \tag{37}$$

The simplest method to realize the condition (37) is to add some number $A$ (for example $A = 1$) to every element of the initial matrix $M$ given by (36) before the Fibonacci coding.

Let us assume now that we have selected the Fibonacci matrix $Q^5$ as the coding matrix, that is

$$Q^5 = \begin{pmatrix} 8 & 5 \\ 5 & 3 \end{pmatrix}. \tag{38}$$

According to Table 2 its inverse matrix is equal

$$Q^{-5} = \begin{pmatrix} -3 & 5 \\ 5 & -8 \end{pmatrix}. \tag{39}$$

Then the Fibonacci coding of the message (36) consists in its multiplication by the "direct" coding matrix (38), that is

$$M \times Q^5 = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \times \begin{pmatrix} 8 & 5 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 8m_1 + 5m_2 & 5m_1 + 3m_2 \\ 8m_3 + 5m_4 & 5m_3 + 3m_4 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} = E, \tag{40}$$

where

$$\begin{aligned} e_1 &= 8m_1 + 5m_2, \\ e_2 &= 5m_1 + 3m_2, \\ e_3 &= 8m_3 + 5m_4, \\ e_4 &= 5m_3 + 3m_4. \end{aligned} \tag{41}$$

Then the code message $E = e_1, e_2, e_3, e_1$ is sent to a channel.

The decoding of the code message $E$ given with (40) is performed in the following manner. The code message $E$ that is represented in the matrix form (40) is multiplied by the inverse matrix (39):

$$\begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \times \begin{pmatrix} -3 & 5 \\ 5 & -8 \end{pmatrix} = \begin{pmatrix} (-3)e_1 + 5e_2 & 5e_1 + (-8)e_2 \\ (-3)e_3 + 5e_4 & 5e_3 + (-8)e_4 \end{pmatrix} = \begin{pmatrix} e'_1 & e'_2 \\ e'_3 & e'_4 \end{pmatrix}. \tag{42}$$

Calculating the elements of the matrix (42) and taking into consideration (41) we get:

$$\begin{pmatrix} e'_1 & e'_2 \\ e'_3 & e'_4 \end{pmatrix} = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} = M.$$

### *6.3. Determinant of the code matrix E*

Let us consider the code matrix $E$ given with the following formula:

$$E = M \times Q_p^n \tag{43}$$

Let us calculate the determinant of the code matrix $E$. According to matrix theory [5] we have

$$\operatorname{Det} E = \operatorname{Det}[M \times Q_p^n] = \operatorname{Det} M \times \operatorname{Det} Q_p^n. \tag{44}$$

Using (29) we can write the formula (44) as follows:

$$\operatorname{Det} E = \operatorname{Det} M \times (-1)^{pn}. \tag{45}$$

Let us formulate this property in the form of the following theorem.

**Theorem 6.** *The determinant of the code matrix E that is got as result of multiplication of the initial matrix M by the coding matrix $Q_p^n$ given by (24) is determined by the determinant of the initial matrix M; in so doing they differ only by a sign: if the product $p \times n$ is even the determinants coincide, in opposite case differ by a sign.*

Note that for the case $p = 1$ formula (45) takes the following form:

$$\operatorname{Det} E = \operatorname{Det} M \times (-1)^n. \tag{46}$$

### 6.4. Connections between the code matrix elements

Let us consider the simplest Fibonacci coding/decoding method for the case $p = 1$. For this case using Table 4 we can write the code matrix $E$ and the initial matrix $M$ as the following:

$$E = M \times Q^n = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \times \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \tag{47}$$

and

$$M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} = E \times Q^{-n} = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \times Q^{-n}. \tag{48}$$

For the case $n = 2k + 1$ we can use the "inverse" matrix (24) as a decoding matrix and then formula (48) takes the following form:

$$\begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \times \begin{pmatrix} -F_{n-1} & F_n \\ F_n & -F_{n+1} \end{pmatrix}. \tag{49}$$

It follows from (49) that the elements of the matrix $M$ can be calculated according to the following formulas:

$$m_1 = -F_{n-1}e_1 + F_n e_2, \tag{50}$$
$$m_2 = F_n e_1 - F_{n-1}e_2, \tag{51}$$
$$m_3 = -F_{n-1}e_3 + F_n e_4, \tag{52}$$
$$m_4 = F_n e_3 - F_{n+1}e_4. \tag{53}$$

Because of the condition (37) we can write the equalities (50) and (51) as the following non-equalities:

$$-F_{n-1}e_1 + F_n e_2 > 0, \tag{54}$$
$$F_n e_1 - F_{n+1}e_2 > 0, \tag{55}$$
$$-F_{n-1}e_3 + F_n e_4 > 0, \tag{56}$$
$$F_n e_3 - F_{n+1}e_4 > 0. \tag{57}$$

The following non-equalities follow from the non-equalities (54) and (55):

$$\frac{F_{n+1}}{F_n} e_2 < e_1 < \frac{F_n}{F_{n-1}} e_2 \tag{58}$$

or

$$\frac{F_{n+1}}{F_n} < \frac{e_1}{e_2} < \frac{F_n}{F_{n-1}}. \tag{59}$$

By analogy we can get from the non-equalities (56) and (57) the following non-equalities:

$$\frac{F_{n+1}}{F_n} e_4 < e_3 < \frac{F_n}{F_{n-1}} e_4 \tag{60}$$

or

$$\frac{F_{n+1}}{F_n} < \frac{e_3}{e_4} < \frac{F_n}{F_{n-1}}. \tag{61}$$

As the ratio of the adjacent Fibonacci numbers strives to the "golden ratio" [1] it follows from (60) and (61) the following approximate equalities that connect the elements of the code matrix (47):

$$e_1 \approx \tau e_2, \tag{62}$$
$$e_3 \approx \tau e_4, \tag{63}$$

where $\tau = \frac{1+\sqrt{5}}{2}$ is the "golden ratio".

By analogy for the case $n = 2k$ using the coding matrix (23) we can write the same approximate equalities (62) and (63) that connect the pairs of the elements $e_1$ and $e_2$, $e_3$ and $e_4$ of the code matrix (47).

Thus, we have found some important mathematical identities (45) and (46) that connect the determinants of the initial matrix $M$ and code matrix $E$ and approximate equalities (62) and (63) that connect the elements of the code matrix (47) for the case $p = 1$. However, for general case of $p$ we can find mathematical identities that connect the code matrix elements similar to the identities (62) and (63).

### 6.5. Error detection and correction

The coding/decoding method considered above gives interesting possibility to detect and correct "errors" in the code message $E$. The main idea of error detection and correction is based on the property of the matrix determinant given by formulas (45) and (46) and on the connections between the code matrix elements given by the approximate equalities (62) and (63) for the case $p = 1$. Strong mathematical relations given by (45), (46), (62) and (63) play a role of "checking relations" of the Fibonacci coding/decoding method.

Let us calculate the determinant of the initial matrix $M$, that is, $\operatorname{Det} M$, and then send it to a communication channel right after the code matrix elements. Note that $\operatorname{Det} M$ plays a role of the "checking element" of the code matrix $E$ received from the communication channel. After receiving the code matrix $E$ and its "checking element" $\operatorname{Det} M$ we can calculate the determinant of the code matrix $E$, that is, $\operatorname{Det} E$, and then compare $\operatorname{Det} M$ to $\operatorname{Det} E$ according to the fundamental "checking relation" (45) and (46) that connect them. If the identity (45) and (46) is true we can conclude that the elements of the code matrix $E$ were transmitted through the communication channel without errors. In the opposite case we have errors in the elements of the code matrix $E$ or in the "checking element" $\operatorname{Det} M$.

Let us show now a possibility of the code matrix restoration after transmission using the properties of the $Q_p$-matrix determinants given by (45) and (46). Let us consider the simplest case of the code matrix given by (47). If after comparison of $\operatorname{Det} M$ to $\operatorname{Det} E$ according to (46) we find that $\operatorname{Det} M$ to $\operatorname{Det} E$ do not fit to the "checking relation" (46) we come to conclusion about possible errors in the elements of the code matrix $E$ or in $\operatorname{Det} M$. Then we can try to correct these errors using the "checking relations" (46), (62) and (63).

Our first hypothesis is that we have the case of "single" error in the code matrix $E$ received from the communication channel. It is clear there are four variants of the "single" errors in the code matrix $E$:

$$\text{(a)} \begin{pmatrix} x & e_2 \\ e_3 & e_4 \end{pmatrix} \quad \text{(b)} \begin{pmatrix} e_1 & y \\ e_3 & e_4 \end{pmatrix} \quad \text{(c)} \begin{pmatrix} e_1 & e_2 \\ z & e_4 \end{pmatrix} \quad \text{(d)} \begin{pmatrix} e_1 & e_2 \\ e_3 & t \end{pmatrix}, \tag{64}$$

where $x$, $y$, $z$, $t$ are possible "destroyed" elements.

In this case we can check different hypotheses (64). For checking the hypothesis (a), (b), (c) and (d) we can write the following algebraic equations based on the "checking relation" (46):

$$xe_4 - e_2e_3 = (-1)^n \operatorname{Det} M (\text{a possible "single error" is in the element } e_1), \tag{65}$$

$$e_1e_4 - ye_3 = (-1)^n \operatorname{Det} M (\text{a possible "single error" is in the element } e_2), \tag{66}$$

$$e_1e_4 - e_2z = (-1)^n \operatorname{Det} M (\text{a possible "single error" is in the element } e_3), \tag{67}$$

$$e_1t - e_2e_3 = (-1)^n \operatorname{Det} M (\text{a possible "single error" is in the element } e_1), \tag{68}$$

where $n$ is an extent of the Fibonacci coding matrix (2).

It follows from (65)–(68) four variants for calculation of the possible "single errors":

$$x = \frac{(-1)^n Det M + e_2e_3}{e_4}, \tag{69}$$

$$y = \frac{-(-1)^n Det M + e_1e_4}{e_3}, \tag{70}$$

$$z = \frac{-(-1)^n Det M + e_1e_4}{e_2}, \tag{71}$$

$$t = \frac{(-1)^n Det M + e_2e_3}{e_1}. \tag{72}$$

The formulas (69)–(72) give four possible variants of "single error" but we have to choice the correct variant only among the cases of the integer solutions $x$, $y$, $z$, $t$; besides, we have to choice such solutions, which satisfies to the additional "checking relations" (62) and (63). If calculations by formulas (69)–(72) do not give an integer result we have to conclude that our hypothesis about "single error" is incorrect or we have "error" in the "checking element" $\operatorname{Det} M''$. For the latter case we can use the approximate equalities (62) and (63) for checking a correctness of the code matrix $E$.

By analogy we can check all hypotheses of "double" errors in the code matrix $E$. As example let us consider the following case of "double errors" in the code matrix $E$

$$\begin{pmatrix} x & y \\ e_3 & e_4 \end{pmatrix}. \tag{73}$$

Using the first "checking relation" (46) we can write the following algebraic equation for the matrix (73):

$$xe_4 - ye_3 = (-1)^n \mathrm{Det}\, M. \tag{74}$$

However, according to the second "checking relation" (62) there is the following relation between $x$ and $y$:

$$x \approx \tau y. \tag{75}$$

It is important to emphasize that Eq. (74) is "Diophantine" one. As the "Diophantine" Eq. (74) has many solutions we have to choice such solutions $x$, $y$, which satisfy to the "checking relation" (75).

By analogy one may prove that using "checking relations" (46), (62) and (63) by means of solution of the "Diophantine" equation similar to (74) we can correct all possible "double errors" in the code matrix $E$.

However, we can show by using such approach there is a possibility to correct all possible "triple" errors in the code matrix $E$, for example

$$\begin{pmatrix} x & y \\ z & e_4 \end{pmatrix}.$$

Thus, our method of error correction is based on verification of different hypotheses about "errors" in the code matrix by using the "checking relations" (46), (62) and (63) and by using the fact that the elements of the code matrix are integers. If all our solutions do not bring to integer solutions it means that the "checking element" $\mathrm{Det}\, M$ is "erroneous" or we have the case of "fourfold error" in the code matrix $E$ and we have to reject the code matrix $E$ as defective and not correctable.

Note it is possible 15 cases of "errors" in the code matrix (47). As our method allows correcting 14 cases among them (all possible "single", "double" and "triple" errors) it means that correctable possibility of the method is equal

$$S_{\mathrm{cor}} = \frac{14}{15} = 0.9333 = 93.33\% \tag{76}$$

### 6.6. Redundancy of the Fibonacci coding method

There are two causes that determine the redundancy of the Fibonacci coding method. The first cause is a "checking element" $\mathrm{Det}\, M$. The second cause is the extent $n$ of the coding matrix (2). As it is shown in [6] the main contribution into a general redundancy gives the "checking element" $\mathrm{Det}\, M$. As is proved in [6] the relative redundancy given by the "checking element" $\mathrm{Det}\, M$ is equal

$$R = 0.333 = 33.3\%. \tag{77}$$

The extent $n$ of the coding matrix (2) influences on two important characteristics of the Fibonacci coding method. For small values of $n$ the redundancy that is contributed by the coding matrix (2) is negligible and a general relative redundancy is determined by (77). However, for the small values of $n$ the equalities (62) and (63) become very approximate that makes worse correctable ability of the Fibonacci coding method. Increasing the extent $n$ makes the equalities (62) and (63) more precise that improves a correctable ability of the Fibonacci coding method. But increasing $n$ brings to increasing of the redundancy. That is why, a problem of choice of the "optimal" value of $n$ is the most important problem of practical application of the Fibonacci coding method.

### 6.7. Comparison of the Fibonacci coding method to the classical coding theory

The error correcting codes are used widely in modern computer systems and computer networks. The main idea of modern algebraic error correcting codes [7] consists in the following. A $n$-bit redundant code combination consists of two groups of bits, $k$ information bits and m checking bits formed from the information bits by modulo 2 addition of certain groups of information bits. A *minimal code distance (Hamming's distance)*, which determines a code ability to detect and correct errors of given multiplicity, is a principal parameter of redundant code.

However, there are one more two important coefficients that determine potential code ability for error detection and correction [7]:

(1) *A potential error detection coefficient* $S_{\mathrm{d}}$, which is determined as a ratio of all detectable passages to all possible passages. This one is expressed through a number $m$ of checking bits as the following:

$$S_{\mathrm{d}} = 1 - 1/2^m. \tag{78}$$

(2) *A potential error correction coefficient* $S_c$, which is determined as a ratio of all correctable passages to all detectable passages. This one is expressed through a number $k$ of information bits as the following:

$$S_c = 1/2^k. \tag{79}$$

As follows from (78) that the coefficient $S_d$ aims for 1 (100%) very quickly when $m$ increases and this fact instills a big optimism about practical application of algebraic redundant codes to detect errors [7]. However, this optimism vanishes when we begin to calculate a potential ability of algebraic codes to correct errors according to (79). In fact, according to (79) a potential correctable ability of redundant code depends on a number $k$ of information bits and it aims for 0 fast when $k$ increases. For example, let us consider (15, 11)-Hamming code ($n = 15$, $k = 11$), which guaranties a correction of all "single error" in the 15-bit code combination. This one uses $m = 15 - 11 = 4$ checking bits. One may calculate that it detects $2^{11}(2^{15} - 2^{11}) = 62{,}914{,}560$ error passages and here it can correct errors only for $2^{15} - 2^{11} = 30{,}720$ cases. Then the ratio $30{,}720/62{,}914{,}560 = 0.0004882$ (0.04882%) gives a value of correctable ability of the given Hamming code.

It is clear that a comparison of $S_c = 0.04882\%$ of the Hamming code to $S_c$ of the Fibonacci coding method given with (76) shows that the Fibonacci coding method exceeds the Hamming code more than 1900 times by potential correctable ability.

However, the main advantage of the Fibonacci coding method consists in the fact that it allows to correct considerably more information units that bits and their combinations. A *matrix element that can be an integer of unlimited value* is a minimal information unit for the Fibonacci coding method.

## 7. Conclusion

The basic idea of the present article is to draw attention to a new class of square matrices called Fibonacci $Q_p$-matrices. These matrices have a unique property given by (29). The formula (29) can be considered as a generalization of the well-known "Cassini formula" given by (4). One may expect that the Fibonacci matrices can find application in theoretical physics.

The Fibonacci coding/decoding method is the main application of the Fibonacci $Q_p$-matrices. This coding method differs from the classical algebraic codes by the following peculiarities:

(1) The Fibonacci coding/decoding method reduces to matrix multiplication, i.e. to well-known algebraic operation, which is realized very well in modern computers.
(2) The main practical peculiarity of the method is the fact that large information units, in particular matrix elements, are objects of error detection and correction. It is proved that the simplest Fibonacci coding method ($p = 1$) can restore with guarantee all "erroneous" code $2 \times 2$ matrices having "single", "double" and "triple" errors.
(3) The correct ability of the method for the simplest case $p = 1$ is equal 93.33% that exceeds essentially all well-known correcting codes.

Of course, the present article is only a brief sketch of a new coding theory based on application of the Fibonacci matrices described in [6]. A practical application of the Fibonacci coding/decoding method demands a deeper comparison of the method to well-known redundant codes by criterions of redundancy and correct ability.

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