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On the Contradiction between the Continuity Axioms (Eudoxus-Archimedes' and Cantor's axioms)

О противоречии между аксиомами непрерывности (Аксиомы Евдокса-Архимеда и Кантора)

> To present an infinite process as a completed, it is impossible without gross violence on the human mind, which rejects such contradictory fantasies.
>
> A.A. Markov, Russian mathematician

Abstract

The article points to the existence of the contradiction between the *continuity axioms* (Eudoxus-Archimedes' and Cantor's axioms), which underlie the classical mathematical theory of measurement, one of the fundamental theory of mathematics. The essence of this contradiction consists in various interpretations of the concept of mathematical infinity in these axioms. The proof of the "*basic equation of measurement*" begins with Eudoxus – Archimedes' axiom, based on the abstraction of *potential infinity*, and ends with Cantor's axiom, based on the abstraction of *actual infinity*. In the process of proving the "basic measurement equation" and other theorems, based on Cantor's set theory, as shown in the works of the prominent Russian mathematician and philosopher Alexander Zenkin (1937 —2006), there is aroused the inconspicuous "jumping over" from the abstraction of *potential infinity* in Eudoxus-Archimedus' axiom to the abstraction of *actual infinity* in Cantor's axiom, that is, the proof of the "*basic measurement equation*" is based on the contradictory axioms what is inadmissible in mathematics. For the first time, the author of the article pointed out on the existence of the contradiction between *continuity axioms* in his book "Introduction to Algorithmic Measurement Theory" (Moscow: Soviet Radio, 1977). Further development of this idea is set forth in author's article *"Is modern mathematics not standing on the "pseudoscientific" foundation?"* (Academy of Trinitarianism, Moscow, El. 77-6567, publication 17034, 11/28/2011). The purpose of this article is to invite mathematicians to discuss this contradiction.

Аннотация

В статье указывается на существование противоречия между аксиомами непрерывности (аксиомы Евдокса-Архимеда и Кантора), которые лежат в основе классической

математической теории измерения, одной из фундаментальных теорий математики. Суть этого противоречия заключается в различных интерпретациях понятия математической бесконечности в этих аксиомах. Доказательство «базового уравнения измерения» начинается с аксиомы Евдокса - Архимеда, основанной на абстракции *потенциальной бесконечности,* и заканчивается аксиомой Кантора, основанной на абстракции *актуальной бесконечности.* В процессе доказательства «основного уравнения измерения» и других теорем, основанных на теории бесконечных множеств Кантора, как показано в работах выдающегося русского математика и философа Александра Зенкина (1937–2006), возникает незаметное «перепрыгивание» от абстракции потенциальной бесконечности в аксиоме Евдокса-Архимеда до абстракции актуальной бесконечности в аксиоме Кантора, то есть, доказательство «основного уравнения измерения» основано на противоречивых аксиомах, что недопустимо в математике. Впервые автор статьи указал на существование противоречия между аксиомами непрерывности в своей книге **«Введение в алгоритмическую теорию измерений»** (Москва: Советское радио, 1977). Дальнейшее развитие этой идеи изложено в статье автора **«Не стоит ли современная математика на «псевдонаучном» фундаменте?»** (Академия тринитаризма, Москва, эл. 77-6567, публикация 17034, 28.11.2011). Цель данной статьи - пригласить математиков для обсуждения этого противоречия.

1. Eudoxus-Archimedes' and Cantor's axioms

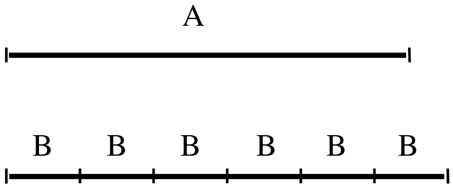
To overcome the first crisis in the foundations of ancient mathematics, associated with the discovery of "*incommensurable segments*", the outstanding geometer Eudoxus proposed the "*method of exhaustion*", by which he built the ingenious theory of relations, underlying the ancient theory of the continuum.

Eudoxus' "exhaustion method" played a prominent role in the development of mathematics. Being a prototype of integral calculus, the "*exhaustion method"* allowed ancient mathematicians to solve problems of calculating volumes of a pyramid, a cone, a ball. In modern mathematics, *Eudoxus' "exhaustion method"* is reflected in the *Eudoxus-Archimedes' axiom*, also called the "*measurement axiom".*

The theory of measurement of geometric quantities, which goes back to "incommensurable segments", is based on a group of axioms, called *continuity axioms* [1], which include Eudoxus – Archimedes' and Cantor's axioms (or Dedekind's axiom).

Eudoxus-Archimedes' axiom ("measurement axiom"): For any two segments A and B (Fig.1) we can find always a positive integer n such that

$$nB>A \tag{1}$$



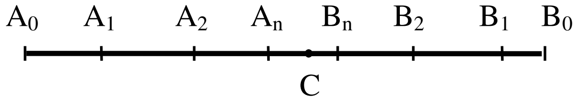


Figure 1. Eudoxus-Archimedes' axiom **Figure 2. Cantor's axiom**

Cantor's axiom (about the "tightening segments"): If an infinite sequence of segments $A_0B_0, A_1B_1, A_2B_2, ..., A_nB_n, ...$, "nested" into each other, is given (Fig. 2), that is, when each segment is part of the previous one, then there is at least one point of *C*, common to all segments.

The main result of the theory of geometric quantities is the proof of the existence and uniqueness of the solution *q* of the "*basic measurement equation*":

$$Q=qV, \tag{2}$$

where *V* is a unit of measurement; *Q* is measurable quantity and *q* is a result of measurement.

Despite the seeming simplicity of the axioms, formulated above, and of the whole mathematical theory of measurement, it is nonetheless a product of more than two thousand years in the development of mathematics and contains a number of deep mathematical ideas and concepts.

First of all, it is necessary to emphasize that *Eudoxus' "exhaustion method"* and the *measurement axiom*, resulting from it (Fig.1), are of practical (empirical) origin; they were borrowed by ancient Greek mathematicians in the practice of measurement. In particular, the "exhaustion method" is a mathematical model for measuring volumes of liquids and bulk solids by "exhaustion"; the measurement axiom, in turn, concentrates thousands of years of human experience, long before the emergence of the axiomatic method in mathematics, billions of times measuring distances, areas and time intervals, and is a concise formulation of the algorithm of measuring of the line segment *A* by using the segment *B* (Fig.1). The essence of this algorithm consists in successively supplementing the segment *B* at the segment *A* and counting the number of segments *B* that are inserted inside the segment *A*. In modern measurement practice, this measurement algorithm is called the "*counting algorithm*".

2. A problem of infinity in mathematics

Cantor's axiom (Fig.2) contains another amazing creation of mathematical thought, an abstraction of *actual infinity*. It is this idea of infinity that underlies Cantor's theory of infinite sets [1].

The idea of *actual infinity* as the main idea of Cantor's (set-theoretic) style of mathematical thinking was strongly criticized by representatives of the so-called *constructive approach* that arose in the 20-th century mathematics in order to overcome the crisis in contemporary mathematics. According to the apt expression of the Russian mathematician A.A. Markov [2], one of the most prominent representatives of the constructive analysis, *"to present an infinite process as a completed it is impossible without gross violence on the human mind, which rejects such contradictory fantasies"*. Earlier, the same idea was expressed in other words by D. Hilbert, who was known for his "finite" representations; has been discussing the concepts of the *finite* and *infinite*, he came to the following conclusion [3]:

"From all our reasoning, we want to make some summary about infinity; the general conclusion is the following: the infinite cannot be realized anywhere. The infinite is not exist in Nature, ann'd this concept is inacceptable as the basis of our rational thinking; here we have a wonderful harmony between being and thinking ... Operating with the infinite can become reliable only through the finite".

The paradoxes or contradictions in Cantor's theory of infinite sets, found at the beginning of the 20-th century, significantly shaken the foundations of mathematics [1]. The various attempts have been made to strengthen them. The most radical of them is the *constructive direction* in the substantiation of mathematics [2], which completely excludes from consideration the abstraction of *actual infinity* and uses a much more "modest" abstraction of infinity, called the abstraction of *potential feasibility*.

To clarify this contradiction, let's consider once again the *"basic measurement equation"* (2). The idea of its proof consists in the following. With the help of the Eudoxus–Archimedes' axiom (Fig.1) and Cantor's axiom (Fig.2), a certain sequence of the *"tightening segments"* $A_0B_0, A_1B_1, A_2B_2, ..., A_nB_n, ...$, which are compared with the *measurable segment* Q, is formed from the *measurement unit* V according to certain rules, called the *measurement algorithm*; when this process goes to infinity, on the basis of Cantor's axiom, for any Q and given V, there will always be found such the *measurable segment* of Q, formed from V, which "absolutely exactly" will be coincided with Q.

The most essential in this proof is a consideration a measurement, as a process that ends during an *actually infinite* time (according to Cantor's axiom). Thus, at the initial stage of the proof of the equation (2) we use the concept of potential infinity (in Eudoxus-Archimedes' axiom) (Fig.1), and at the concluding stage we “jumping over” through this concept and use the concept of *actual infinity* (in Cantor’s axiom) (Fig.2).

Thus, the “jumping over” from the abstraction of potential infinity (in the Eudoxus-Archimedes’ axiom) at the initial stage of the proof of the equation (2) to the abstraction of actual infinity (in Cantor's axiom) at the concluding stage of the proof of the equation (2), imperceptible at first glance, is the basic idea of the proof of the “basic measurement equation” (2). Without this “jumping over”, the proof of the “basic equation of measurement” (2) becomes “meaningless”.

The author of this article came to this idea back in 1972 in his doctoral dissertation **“Synthesis of optimal algorithms for analog-digital conversion”** and he outlined this idea in his 1977 book, **Introduction to Algorithmic Measurement Theory** [1] as follows:

“It is appropriate to pay attention to the internal contradictoriness (in the dialectical sense) of the set-theoretic theory of measurement (and as a consequence of the theory of real numbers), which allows in its initial positions (continuity axiom) the coexistence of dialectically contradictory ideas about infinity - actual infinity in Cantor’s axiom (and Dedekind’s axiom) and potential, "becoming", unfinished infinity - in Archimedes’ axiom".

That is, the existing classical theory of measurement and the arising from it theory of real numbers, based on Cantor's axiom, are internally contradictory, but such theories cannot be the basis of mathematics! Otherwise, all mathematics in the whole becomes an internally contradictory theory. This really had happened in mathematics at the beginning of the 20-th century, when the contradictions were found in Cantor’s theory of infinite sets.

3. Criticism of Cantor’s theory of sets

Cantor's theory of infinite sets caused a storm of protests already in the 19-th century. A detailed analysis of the criticism of this theory was given in the Chapter IX “Paradise Barred: A New Crisis of Reason” of the remarkable book by the prominent American historian of mathematics Morris Kline (1908 –1992) “Mathematics. Loss of Certainty " [4].

Many famous mathematicians of the 19-th century spoke out sharply negatively about this theory. Leonid Kronecker (1823-1891), who had a personal dislike to Cantor, called him a *charlatan.* Henri Poincare (1854-1912) called the theory of sets a “serious illness” and considered it as a kind of “mathematical pathology”. In 1908, he declared: *"The coming generations will regard the theory of sets as a disease from which they have recovered."*

Unfortunately, Cantor’s theory had not only opponents, but also supporters. The British philosopher Bertrand Arthur William Russell (1872–1970) called Cantor as one of the great thinkers of the 19-th century. In 1910, Russell wrote: *"Solving problems that have long enveloped mystery in mathematical infinity is probably the greatest achievement that our age should be proud of."*

In his speech at the 1-st International Congress of Mathematicians in Zurich (1897), the famous mathematician Hadamard (1865-1963) emphasized that the main attractive feature of Cantor's set theory is that for the first time in mathematical history, the classification of sets, based on the concept of "cardinal number", was given. In his opinion, the amazing mathematical results, which follow from Cantor's set theory, should inspire mathematicians to new discoveries. Thus, in Hadamard's speech, Cantor's theory of infinite sets was elevated to the level of the main mathematical theory, which can become the foundation of mathematics.

Different points of view on Cantor's theory of infinite sets are set forth in Wikipedia article [5]. Let's cite some critical remarks addressed to Cantor's theory of infinite sets, taken from the article [5].

"*Initially, Cantor's theory was controversial among mathematicians and (later) philosophers. As Leopold Kronecker claimed: "I don't know what predominates in Cantor's theory – philosophy or theology, but I am sure that there is no mathematics there"… Many mathematicians agreed with Kronecker that the completed infinite may be part of philosophy or theology, but that it has no proper place in mathematics... Mathematician Solomon Feferman has referred to Cantor's theories as "simply not relevant to everyday mathematics."*

The author of the present article completely shares the point of view Leopold Kronecker and Solomon Feferman.

4. Research of Alexander Zenkin

In recent years, in the works of the outstanding Russian mathematician and philosopher Alexander Zenkin (1937-2006) [6-8], the radical attempts had been made to "purify" mathematics from Cantor's theory of sets, based on conception of *actual infinity*.

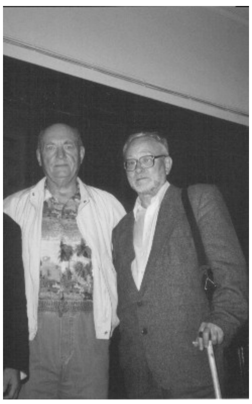


Figure 3. Alexey Stakhov and Alexander Zenkin

(Moscow University, April 29, 2003: Stakhov's lecture **"A new type of elementary mathematics and computer science based on the Golden Section",** delivered at the joint meeting of the seminar "Geometry and Physics", Department of Theoretical Physics, Moscow University, and Interdisciplinary Seminar "Symmetries in Science and Art" of the Institute of Mechanical Engineering, Russian Academy of Sciences)

The analysis of Cantor's theory of infinite sets, presented in [6-8], led Alexander Zenkin to the conclusion that the proofs of many Cantor's theorems are logically incorrect, and the whole "Cantor's theory" is in a certain sense the "19-th century mathematical hoax." The mathematicians of the 19-th century were fascinated by Cantor's theory, and, by accepting his unusual theory without proper critical analysis, elevated it to the rank of the greatest mathematical discovery of the 19-th century and laid it at the foundations of mathematics.

By the way, in the Wikipedia article [5] there are the links to two Zenkin's critical articles [7,8], concerning Cantor's theory of infinite sets.

Conclusion

1. The main goal of the present article is to give a comparative analysis of two *continuity axioms* (Eudoxus-Archimedes' and Cantor's axioms). Only with such an analysis we can find the fundamental contradiction between them. The essence of the contradiction is reduced to different interpretations of *infinity concept*, used in mathematics: *actual infinity* in Cantor's axiom and *potential infinity* in Eudoxus-Archimedes' axiom. The author came to this idea back in 1977 while writing the book "Introduction to algorithmic measurement theory" [1]. This book is written in Russian and had not been translated into English.

The establishment of paradoxes in Cantor's theory of infinite sets considerably cooled the enthusiasm of mathematicians to this theory. The Russian mathematician and philosopher Alexander Zenkin put the final point in the critical analysis of Cantor's theory [6-8]. He showed that the main Cantor's error was the adoption of the abstraction of *actual infinity* what is, according to Aristotle, is unacceptable in mathematics. But without the abstraction of *actual infinity*, Cantor's theory of infinite sets is untenable! As it is mentioned, Aristotle was the first great thinker who drew attention to this problem and warned about the impossibility of using the concept of *actual infinity* in science and mathematics. Aristotle's saying **"Infinitum Actu Non Datur"** is widely known in the history of science and mathematics and underlies the basis of *constructive mathematics* [2].

2. In order to eliminate the above contradiction, the author of this article had removed Cantor's axiom from measurement theory and constructed a new theory of measurements, *algorithmic measurement theory,* based on the abstraction of *potential infinity* [1]. In author's article [6] the following question was posed: *"Is modern mathematics not standing on the "pseudo-scientific" fundament?"* Thus, the author's *algorithmic measurement theory* is the example of *constructive* mathematical theory that negates the use of the *actual infinity* abstraction as an internally contradictory concept [2].

3. It should be noted that the critical information, related to Cantor's theory of infinite sets and described in Kline's book [4] and Wikipedia paper [5], calls us to searching the answer to the question, put by the author in the article [9]. Note, that Wikipedia paper [5] discusses different points of view on Cantor's theory of infinite sets (Leopold Kronecker, Solomon Feferman, Alexander Zenkin and others).

4. Author's *algorithmic measurement theory* [1], created without the usage of Cantor's axiom and without conception of *actual infinity*, is possibly one of the first variant of mathematical measurement theory, which leads us to the important mathematical and applied results, having great significance for the development of mathematics and computer science. This theory is a source for the following mathematical theories, published in the famous mathematical journals and Publishing Houses:

4.1. Theory of the *optimal measurement algorithms* [1]

4.2. Theory of the *numeral systems with irrational bases* [10-12]

4.3. The ternary *mirror-symmetrical arithmetic* [13]

4.4. The *"golden" hyperbolic Fibonacci and Lucas functions* [14 - 16]

4.5. The *Mathematics of Harmony* as a new direction in mathematics and computer science [17]

4.6. The "*Golden" Non-Euclidean Geometry* as a new direction in Non-Euclidean Geometry [18]

4.7. Other original papers in this area [19-24], published in *British Journal of Mathematics & Computer Science* and *Physical Science International Journal.*

5. The main conclusion from this article is that we must use Cantor's theory of infinite sets in modern mathematics with great care and follow to the famous Aristotle saying **"Infinitum Actu Non Datur".**

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