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THE ANALYSIS OF NON-LINEAR SYSTEMS - THE OSCILLATING AND ROTATING MATHEMATICAL PENDULUM BY MEANS OF THE SIZED SCALING METHOD

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Abstract

It is shown that even for such non-linear system, as the mathematical (simple) pendulum (MP) [1,2], known more than 350 years, it is possible to receive by means of the method of sized scaling (SS) a row of new nontrivial results. In the SS method along with well-known scale for the oscillation frequency $M_0 = \sqrt{g/L}$ (g is the free fall acceleration, L is the MP length) may be introduced, entering from a basic scales $M_1 = V/L$ (V is the bob speed in the lower point of a path) and $M_{-1} = g/L$, $M_0 = \sqrt{M_1 \cdot M_{-1}}$, and then as there geometric averages the infinite set of other scales. At the same time, using parameters κ and $k = 1/\kappa$ in the elliptic functions and integrals ($\kappa^2 = V^2/4gL$ is equal to the relation of the kinetic energy of the MP bob in the lower point of a path to the maximum potential energy in the upper point), it is possible to describe the dynamics both of the oscillating and rotating MP in spite of the fact that it is topologically different movements. It is essential also that for the MP it is found a number of exact and interesting ratios which are expressed through the gold ratio constants $\phi = (-1+\sqrt{5})/2$ and $\Phi = (1+\sqrt{5})/2$. Besides, it is shown that basic scales can form the so-called Kepler's and meta-triangles found, in particular, in geometry of the Great Pyramid of Cheops.

The MP (also called the simple pendulum) is a system consisting of a rod of length L and of negligible mass or a string, which also is assumed to be massless and unstrechable, and a point mass m, attached to the rod or string and called the pendulum bob. The rod or string is attached to a pivot which is the point to swing from.

The forces acting on the bob are the tension in the string $\vec{F}$ and the gravitational force $\vec{P}=m\vec{g}$. When the bob is displaced and released the tangential component of $\vec{P}$ produces a restoring force, which always acts in the direction opposite to the displacement of the bob. As a result the bob oscillates. However oscillations are not isochronous and only at very small initial deflection angles of a rod or string from a vertical ($\varphi_0 \simeq 0$) the oscillation period T practically does not depend on oscillation amplitude: $T=2\pi\sqrt{L/g}\cdot(1+\varphi_0^2/16+11\cdot\varphi_0^4/3072+\cdots)\simeq 2\pi\sqrt{L/g}$ (g - is the free fall acceleration).

The dependence of the period of the MP on the free fall acceleration forms the basis of a very accurate method for determining this acceleration. This method is widely used in practice.

In this article a row of new nontrivial pequliarities in the dynamics of the oscillating and rotating mathematical MP is set with the help of the method of the sized scaling (SS), which was offered by us in [3,4].

Assuming that the expression for the MP oscillation frequency $\omega=2\pi/T$ depends on 3 possible parameters $\omega\sim g^{\alpha}\cdot L^{\beta}\cdot V^{\gamma}\cdot m^{0}$ (V is the bob speed in the lower point of a path) we receive the equation for determination of 3 (!) indexes α,β,γ through 2 (!) values (units of time T and lengths L, $[V]=L\cdot T^{-1}$): $T^{-1}=L^{\alpha}\cdot T^{-2\alpha}\cdot L^{\beta}\cdot L^{\gamma}\cdot T^{-\gamma}$. Mass of the bob is not included into this equation ($m^{\delta}=m^{0}$) since all bodies have the same gravitational acceleration.

From this equation it follows that there is an infinite number of combinations of indexes determined by expressions $\alpha=(1-\gamma)/2$, $\beta=(-1-\gamma)/2, -\infty<\gamma<+\infty$. The values of indexes α,β,γ lie in the plane $1\cdot\alpha+1\cdot\beta+1\cdot\gamma=0$.

Thus, along with the well-known scale of the MP oscillation and rotation frequencies $M_0=\sqrt{g/L}$ ($\gamma=0$) exists infinitely large number of other scales $M_{\gamma}=(2\kappa)^{\gamma}\sqrt{g/L}$ ($\kappa^2=V^2/4gL$ is the relation of the bob kinetic energy in the lower point of a path $mV^2/2$ to the maximum potential energy $2mgL$ in the upper point), γ- any number. So, in case of $\gamma = 0$ we receive "usual" scale $M_0 = \sqrt{g/L}$, in case of $\gamma = \pm 1$ the basic scales $M_1 = V/L$, $M_{-1} = g/V$ from 2 parameters, in case of $\gamma = \pm 1/2, \pm 1/3$ the scales from 3 parameters $M_{1/2} = \sqrt{\sqrt{g/L} \cdot (V/L)}$, $M_{-1/2} = \sqrt{\sqrt{g/L} \cdot (g/V)}$, $M_{1/3} = \sqrt[3]{(g/L) \cdot (V/L)}$, $M_{-1/3} = \sqrt[3]{(g/L) \cdot (g/V)}$.

It is essential that both the geometric averages from 2 symmetric scales $\sqrt{M_{-\gamma} \cdot M_\gamma} = M_0$ and the generalized geometric averages from $2n+1$ symmetric scales $\sqrt[2n+1]{\prod_{i=-n}^{i=n} M_i} = M_0$ are equal to scale $M_0 = \sqrt{g/L}$.

From the conservation law of a total energy for the conservative system – the MP

$$m\frac{L^2 (d\varphi/dt)^2}{2} + mgL(1 - \cos\varphi) = E \qquad (1)$$

it follows that the period of oscillations of the MP is defined by expression:

$$T_{OS} = 4\sqrt{\frac{L}{2g}} \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}} = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \kappa^2 \cdot \sin^2\theta}} = 4\sqrt{\frac{L}{g}} \cdot K(\kappa) \qquad (2)$$

φ- angle of deflection of the rod (string) from vertical, $\sin\theta = \sin(\varphi/2)/\sin(\varphi_0/2)$, $\kappa^2 = \sin^2(\varphi_0/2) = V^2/4gL$, $K(\kappa) = \int_0^{\pi/2} d\theta / \sqrt{1 - \kappa^2 \sin^2\theta}$ is the 1st kind full elliptic integral.

At $0 \le \kappa < 1$ for approximation of $K(\kappa)$ integral the following power series are used [5-7]:

$$K(\kappa) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \kappa^2 \sin^2\theta}} = \frac{\pi}{2}\{1 + \sum_{n=1}^{\infty} [\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n}]^2 \kappa^{2n}\} \qquad (3)$$

At $\kappa \simeq 1$ we used another approximation of $K(\kappa)$ integral:

$$K(\kappa) = \Lambda + \frac{\Lambda - 1}{4}\kappa_1^2 + \frac{9}{64}(\Lambda - \frac{7}{6})\kappa_1^4 + \frac{5}{256}(\Lambda - \frac{37}{30})\kappa_1^6 + \ldots \qquad (4),$$

where $\Lambda = \ln(4/\kappa_1)$, $\kappa_1 = \sqrt{1 - \kappa^2}$.

From the law of conservation of energy for the rotating MP:

$$mv^2(\varphi)/2 = m(Ld\varphi/dt)^2/2 = mV^2/2 - mgL(1-\cos\varphi) \qquad (5)$$

it follows that the period of rotation is defined by expression:

$$T_{RT} = 2\frac{2L}{V}\cdot\int_0^{\pi/2}\frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} = \frac{4L}{V}\cdot K(k) \qquad (6)$$

where $\vartheta = \varphi/2$, $k^2 = mg2L/(mV^2/2) = 4gL/V^2 = 1/\kappa^2$.

Proceeding from (2), (6), it is possible to show that the frequencies in different scales are defined by the following expressions for the oscillating and rotating MP:

$$\omega_{\gamma OS} = (\pi/2)(2\kappa)^{-\gamma}K^{-1}(\kappa) \qquad (7)$$

$$\omega_{\gamma RT} = (\pi/2)(2\kappa)^{1-\gamma}K^{-1}(1/\kappa) \qquad (8)$$

In the case of the inversion of parameters $\kappa \leftrightarrow 1/\kappa$ the elliptic functions for the oscillating MP $K(\kappa)$ are replaced with the functions for the rotating MP $K(k=1/\kappa)$ and vice versa. In this regard it should be pointed out, for example, that at inversion $\kappa \leftrightarrow 1/\kappa$ in the scale $M_{1/2} = \sqrt{\sqrt{g/L}\cdot(V/L)}$ which is geometric average for the scales $M_0 = \sqrt{g/L}$ and $M_1 = V/L$ corresponding accordingly to the oscillating and to rotating movements the rotation frequency is exactly twice more larger than the oscillation frequency: $\omega_{RT} = 2\omega_{OS}$ (see fig. 1).

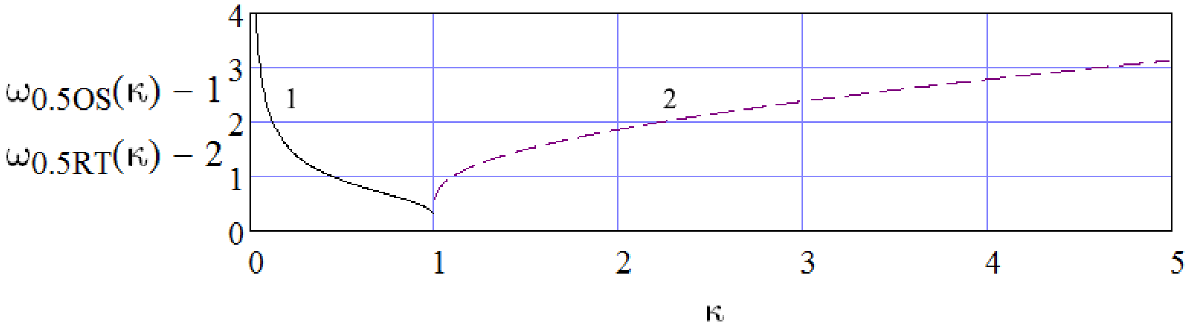

Fig.1. Calculated dependences of the MP oscillation and rotation frequencies $\omega_{0.5OS}(\kappa) = (\pi/2)(2\kappa)^{-1/2}K^{-1}(\kappa)$ and $\omega_{0.5RT}(\kappa) = (\pi/2)(2\kappa)^{1/2}K^{-1}(1/\kappa)$ at $\gamma = 1/2$ (curves 1 ($0 \le \kappa \le 1$), 2 ($1 \le \kappa$) respectively), $\omega_{0.5RT}(1/\kappa)/\omega_{0.5OS}(\kappa) = 2$.

This symmetry takes place and in the MP equations of motion:

$$v_{OS}(t) = (Ld\varphi/dt)_{OS} = V \cdot cn(t \cdot \sqrt{g/L}, \kappa), \quad 0 \le \kappa \le 1 \qquad (9),$$

$$v_{RT}(t) = (Ld\varphi/dt)_{RT} = V \cdot dn(t \cdot V/2L, \kappa), \quad 1 \le \kappa < \infty \qquad (10)$$

At inversion of parameters $\kappa \leftrightarrow 1/\kappa$ the elliptic function cn for the oscillating MP are replaced with dn function for the rotating MP and vice versa.

At the same time elliptic functions: cosine - cn and sine - sn are defined as follows. According to (2) dimensionless time τ of the bob movement from vertical position to the deviation angle θ it is equal:

$$\tau = t \cdot \sqrt{g/L} = \int_0^{\theta} d\theta / \sqrt{1-\kappa^2 \sin^2\theta} \qquad (11)$$

The upper limit in this integral θ is the inverse function of this integral and is called “amplitude” and designated as $\theta = am(\tau, \kappa^2)$.

Functions $\sin(am(\tau,\kappa^2) = sn(\tau,\kappa)$, $\cos(am(\tau,\kappa^2) = cn(\tau,\kappa)$ are called the elliptic sine and cosine, $sn^2(\tau) + cn^2(\tau) = 1$, $sn(-\tau) = -sn(\tau)$, $cn(-\tau) = cn(\tau)$.

Their main periods are $4K$ and $2K' \cdot i$, where $K' = K(k')$, $k' = \sqrt{1-k^2}$, $i = \sqrt{-1}$ [6-7].

The elliptic function dn is defined as $dn(\tau,k) = \sqrt{1-k^2 sn^2(\tau,k)}$ and has the main periods $2K(k)$, $2K'i$ [6-7].

From definitions of the elliptic functions it can be found the following transformations at parameters inversion ($\kappa \leftrightarrow 1/\kappa$): $sn(\tau,1/k) = k \cdot sn(\tau/k)$, $cn(\tau,1/k) = dn(\tau,1/k)$, $dn(\tau,1/k) = cn(\tau/k,k)$. And, respectivelym at this transformations the scale $\sqrt{g/L}$ for time changes on $\sqrt{g/L}/k = V/2L$.

For the movement on a separatrix ($\kappa^2 = k^2 = 1$) equation (1) has exact solutions:

$$d\varphi/dt = \pm 2\sqrt{g/L}\cdot\cos(\varphi/2) = \pm 2\sqrt{g/L}/ch(\sqrt{g/L}\cdot t) \qquad (12),$$

$$\varphi = 2\cdot\arcsin(th(\sqrt{g/L}\cdot t)) = 4\cdot arctg(\exp(\sqrt{g/L}\cdot t)) - \pi \qquad (13)$$

It is found that the dependences $\omega_\gamma(\kappa^2)$ can be or the monotonic, or having a maximum in the field of oscillations $(0 \le \kappa^2 \le 1)$ or in the field of rotation $(1 \le \kappa^2 < \infty)$. At the same time all dependences $\omega_\gamma(\kappa)$ are crossed in one point in case of any γ (see fig. 2): $\omega_\gamma(\kappa = 0{,}5)/2\pi \simeq 0{,}1483 \simeq 1/(\phi+\Phi)(\phi^2+\Phi^2)$.

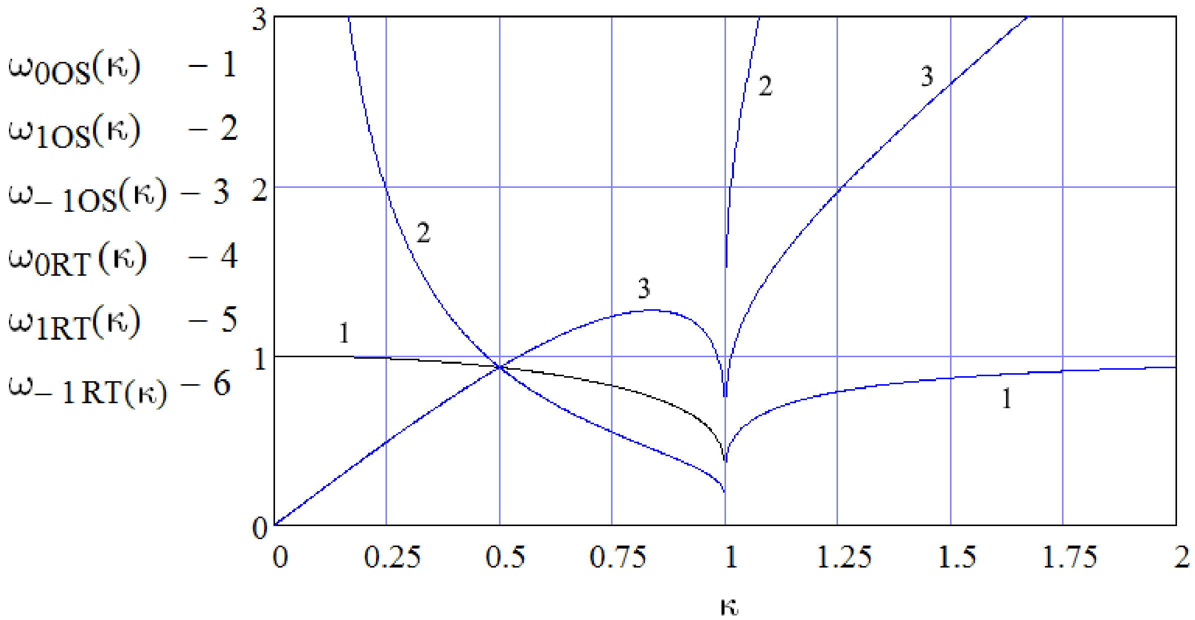

Fig. 2. Calculated dependences of the MP oscillation and rotation frequencies $\omega_{\gamma OS}(\kappa)$ and $\omega_{\gamma RT}(\kappa)$ at various values of parameter γ: $\gamma = 0$ (scale $\sqrt{g/L}$), $\gamma = 1$ (scale V/L). $\gamma = -1$ (scale g/V) - curves 1, 2, 3. 4, 5, 6 respectively.

It is established that the maximum in dependences $\omega_\gamma(\kappa)$ takes place at $\gamma < 0$. For the curve 3 in fig. 2 $\omega_{-1OS}(\kappa)_{max}/2\pi = \omega_{-1OS}(0{,}83746)/2\pi \simeq (\phi+\Phi)^{-2}$!! At the same time $\kappa^2_{max} \simeq 0{,}70134 \simeq \sqrt{3}\Phi/4$. Accounting that $\kappa^2 = V^2/4gL$, we can find that the optimal MP length L_{opt} for scale $M_{-1} = g/L$, at which the dependence $\omega_\gamma(\kappa)$ is the most flat, is equal $L_{opt} \simeq V^2/\sqrt{3}\Phi g = V^2/\sqrt{1+\Phi^4}\, g$.

The existence of a maximum in dependence $\omega_\gamma(\kappa)$ for the scales having V in denominator ($\gamma < 0$) it is possible to explain as follows. At small κ^2 (small V) the scale M_γ is great. With increase of κ^2 the frequencies in the beginning grow because of decrease of scale M_γ. However at further increase of κ^2 effects sharp decrease of $K^{-1}(\kappa^2)$, that is caused by the fact that at larger V the oscillating bob gets to the area of angles $\varphi \simeq 180^o$ at which movement becomes very slow and $\omega_\gamma(\kappa=1) \simeq 0$ (period $T_\gamma(\kappa=1) \to \infty$). This reasons explain also that for the scales having V in numerator ($\gamma > 0$) with the increase of κ^2 takes place the monotonic decrease of $\omega_\gamma(\kappa)$ (curve 2 in fig. 2).

It is important that the maximum in dependences $\omega_\gamma(\kappa)$ at $\gamma < 0$ is minimum at $\gamma \simeq 0{,}203643 \simeq (\phi+\Phi)/11 = (\phi+\Phi)/(\Phi^5 - \phi^5) \simeq 0{,}203278$ and at the same time coincides with a cross point of all schedules (!) at $\kappa^2 = 1/4$ (see fig. 3).

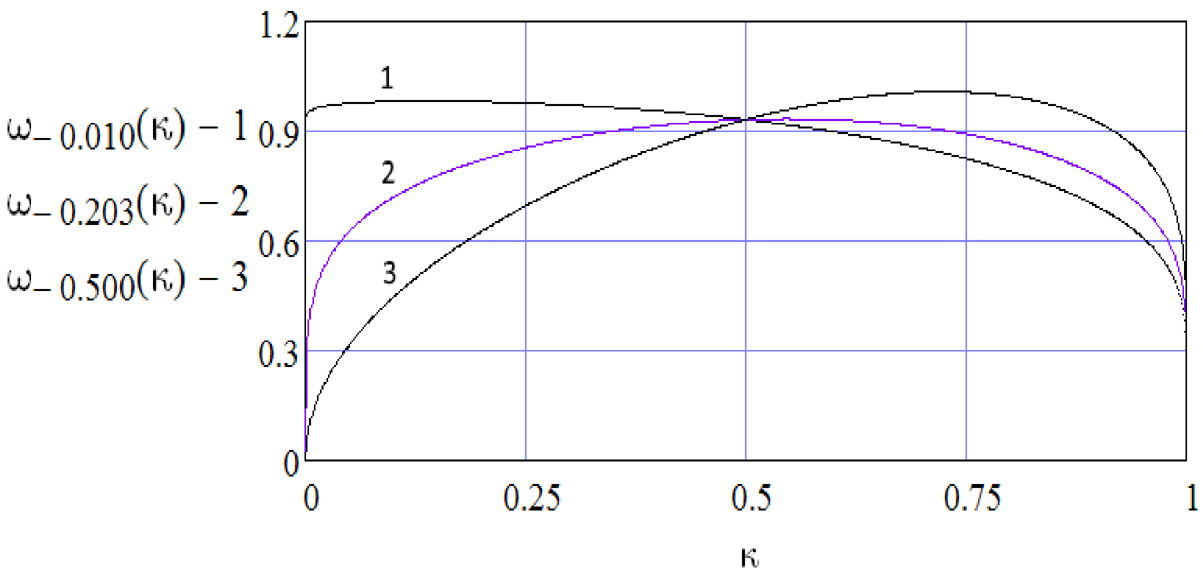

Fig. 3 Calculated dependences of the MP oscillation frequencies $\omega_\gamma(\kappa)$ at $\gamma = -0{,}01$, $\gamma = -0{,}203643790$, $\gamma = -0{,}50$ (curves 1, 2, 3 – respectively).

In this case $\omega_\gamma(\kappa)/2\pi_{max} = \omega_{-2036}(0{,}5)/2\pi \simeq 0{,}1483 \simeq 1/(\phi+\Phi)(\phi^2+\Phi^2)$, $\kappa^2 = 1/4 = V^2/4gL_{opt}$ and, therefore, the optimal length is $L_{opt} = V^2/g$!

It is found that at $\kappa^2 = 1/4$ geometric averages $\sqrt{M_{-\gamma} \cdot M_\gamma}$ and $\sqrt{\omega_{-\gamma} \cdot \omega_\gamma}$ are equal to their arithmetic averages $(M_{-\gamma} + M_\gamma)/2$ and $(\omega_{-\gamma} + \omega_\gamma)/2$, as a result all scales at $\kappa^2 = 1/4$ become equal.

Corresponding to equality of scales at $\kappa^2 = 1/4$ in this case the angle of the maximal deviation of the MP string (or rod) $\varphi_0 = \pi/3 = \pi/(\phi^2 + \Phi^2)$, at the same time the MP in any scale ($-\infty < \gamma < +\infty$) gives the same results !

Let's compare to scales $M_\gamma, M_0, M_{-\gamma}$ the line segments entered in a semicircle as it is shown in fig. 4. At change κ^2 the line segment CD moves on the diameter AB and at $\kappa^2 = 1/4$ bisects AB and at this time $M_0 = M_\gamma = M_{-\gamma}$.

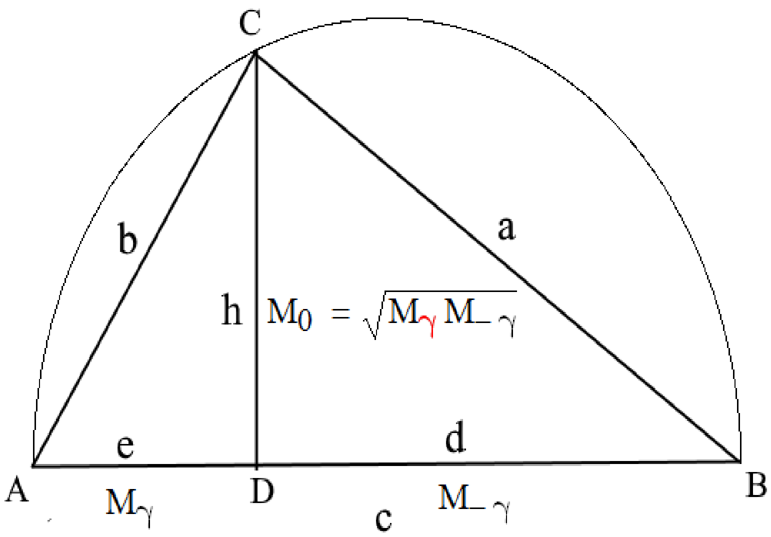

Fig. 4

At $\kappa^2 = 1/4$ angles A, B in fig. 1 are equal $\pi/4$ for any scales. At the same time when passing value $\kappa = 1/2$ it can be sharp change of angles A, B from 0 to $\pi/2$ or from $\pi/2$ to 0. The corresponding dependences $\angle A(\kappa)$, $\angle B(\kappa)$ are shown in fig. 5 for $\gamma = \pm 0{,}2,\ \pm 1,\ \pm 12$ (curves 1-6 respectively).

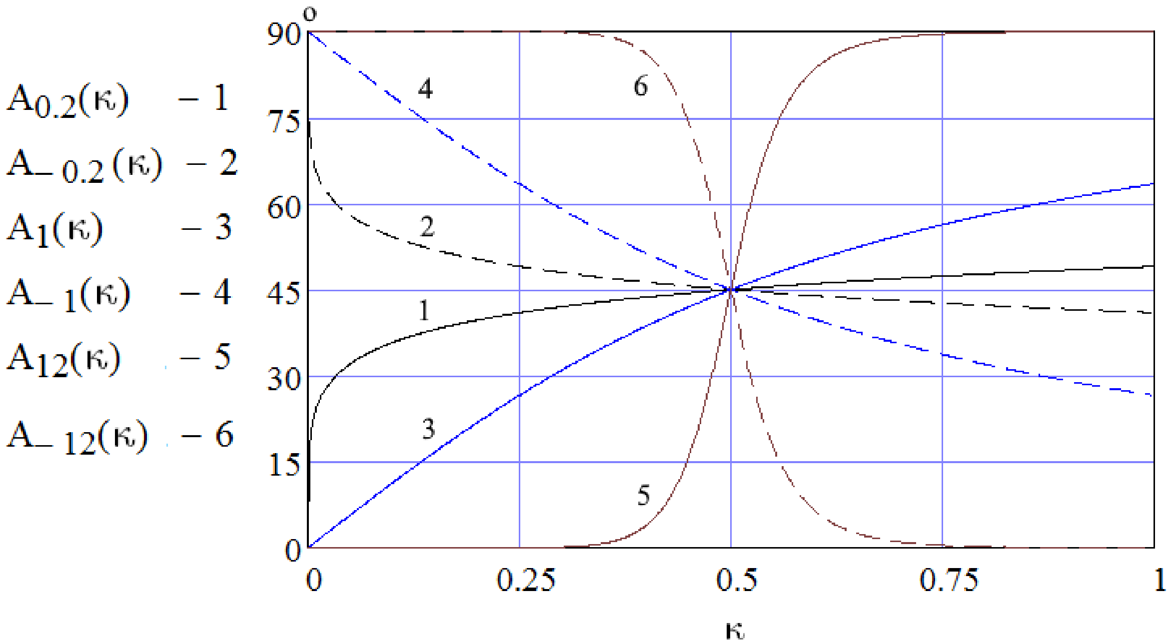

Fig. 5

Thus, at $\kappa = 1/2$ in the MP it can be the effect of “bistability of scales”: the ratio $M_{-\gamma} \gg M_{\gamma}$ changes on inverse $M_{\gamma} \gg M_{-\gamma}$.

Accounting that $M_{\gamma} = (2\kappa)^{\gamma}\sqrt{g/L}$, $M_{-\gamma} = (2\kappa)^{\gamma}\sqrt{g/}$, $M_0 = \sqrt{g/L}$ we receive that in fig. 4 $\angle A = \mathrm{arctg}(2k)^{-\gamma}$, $\angle B = \mathrm{arctg}(2k)^{\gamma} = \pi/2 - \angle A$. In case $\gamma = 1$ $M_1 = 2\kappa\sqrt{g/L} = V/L$, $M_{-1} = (1/2\kappa)\sqrt{g/L} = g/V$ ($\kappa = V/2\sqrt{gL}$).

In [8] a certain abstract rectangular triangle $\triangle ABC$ with the sides $AB = c$, $AC = b$, $BC = a$ (see fig. 4) is called Kepler's triangle, if lengths a, b, c form a geometrical progression. If $CD = h$ is the perpendicular to the hypotenuse AB, $AD = e$, $BD = d$, then in Kepler's triangle $h = \sqrt{d \cdot e}$, $a = \sqrt{c \cdot d}$, $b = \sqrt{c \cdot e}$, $h \cdot c = a \cdot b$, $h^{-2} = a^{-2} + b^{-2}$.

In [9] the rectangular triangle is called a meta-triangle if, along with the Pythagorean theorem $a^2 + b^2 = c^2$, the ratio $a \cdot b = c$ is executed.

In [8] it is supposed that the meta-triangle is a special case of the Kepler’s triangle, in [9], on the contrary, it is supposed that the Kepler's triangle is a special case of a meta-triangle.

These two types of triangles have been found earlier in our work [4] without using the ambitious names. In this article we will give for these triangles concrete physical sense according to designations in fig. 4.

Assuming that $\gamma=1$, $h=M_0=\sqrt{g/L}=1$, we will consider four important cases of relations between the pieces AD and BD. As a result we will receive a number of interesting geometrical and physical relationships.

1. $AD/BD=M_1/M_{-1}=(2\kappa)^2=V^2/gL=\phi^2$. From the energy conservation law $mV^2/2=mgL(1-\cos\varphi_M)$ we will receive that the angle of maximum deviation of the MP string (or rod) $\varphi_M=\arccos(1-\phi^2/2)=\pi/5=\pi/(\phi+\Phi)^2$. At the same time $AD=M_1=\phi$, $BD=M_{-1}=\Phi$, $CD=M_0=1$, $AC=\sqrt{\phi^2+1}$, $BC=\sqrt{\Phi^2+1}$, $\angle A=\mathrm{arctg}\Phi\simeq 58{,}282\,525\,589^{o}$, $\angle B=\mathrm{arctg}\phi\simeq 31{,}717\,474\,411^{o}$.

It is interesting that in this case “der wurf” (german - throw) for the pieces $AD=e, CD=h, BD=d$ is eqial to the ideal value $\Phi^2/2=\phi+1/2$ [10]:

$$W_{ehd}=\frac{(e+h)(h+d)}{h(e+h+d)}=\Phi^2/2\simeq 1{,}309\,016\,994$$

But this $\triangle ABC$ is not Kepler's triangle and at the same time it is meta-triangle (!):

$$c/a=(\phi+\Phi)/\sqrt{\Phi^2+1}\neq a/b=\sqrt{\Phi^2+1}/\sqrt{\phi^2+1},$$

$$a\cdot b=\sqrt{1+\Phi^2}\cdot\sqrt{1+\phi^2}=c=\phi+\Phi$$

1. 2. $AD/BD=M_1/M_{-1}=(2\kappa)^2=V^2/gL=\Phi^2$. In a similar manner we will receive $\varphi_M=\arccos(1-\Phi^2/2)=3\pi/5=\pi(\phi^2+\phi^2)/(\phi+\Phi)^2$, $AD=M_1=\Phi$,

2. $BD=M_{-1}=\phi$, $CD=M_0=1$, $AC=\sqrt{\Phi^2+1}$, $BC=\sqrt{\phi^2+1}$, $\angle A=\mathrm{arctg}\phi$,

3. $\angle B=\mathrm{arctg}\Phi$, $W_{dhe}=\Phi^2/2$.

This $\triangle ABC$ is also not Kepler's triangle, but it is also meta-triangle as

$$c/b=(\phi+\Phi)/\sqrt{\Phi^2+1}\neq b/a=\sqrt{\Phi^2+1}/\sqrt{\phi^2+1},$$

$$a\cdot b=\sqrt{1+\phi^2}\cdot\sqrt{1+\Phi^2}=c=\phi+\Phi$$

4. 3. $AD/BD = M_1/M_{-1} = (2\kappa)^2 = V^2/gL = \phi$. Also as in variant 1 we will receive that $\varphi_M = \arccos(1-\phi/2) \simeq 0{,}807\,483\,293 \simeq \Phi/2 \leftrightarrow 46{,}292\,025\,263^o$,

$AD = M_1 = \sqrt{\phi}$, $BD = M_{-1} = \sqrt{\Phi}$, $CD = M_0 = 1$, $AC = \sqrt{1+\phi}$, $BC = \Phi$,

5. $\angle A = \mathrm{arctg}(\sqrt{\Phi}) \simeq 51{,}827\,292\,373^o$, $\angle B = \mathrm{arctg}\sqrt{\phi} = 38{,}172\,707\,627^o$,

6. $$W_{ehd} = \frac{(\sqrt{\phi}+1)(1+\sqrt{\Phi})}{1\cdot(\sqrt{\phi}+1+\sqrt{\Phi})} = 1 + \frac{1}{1+\sqrt{\phi}+\sqrt{\Phi}} \simeq 1{,}326\,992\,830 \neq \Phi^2/2$$

7. But this $_\triangle ABC$ is simultaneously the Kepler's and meta-triangle (!!) as

$$c/a = (\sqrt{\phi}+\sqrt{\Phi})/\Phi = a/b = \Phi/\sqrt{1+\phi} = \sqrt{\Phi} \simeq 1{,}272\,019\,650,$$

$$a\cdot b = \Phi\cdot\sqrt{\Phi} = c = \sqrt{\phi}+\sqrt{\Phi} \simeq 2{,}058\,171\,027$$

4. $AD/BD = M_1/M_{-1} = (2\kappa)^2 = V^2/gL = \Phi$. Also as in variant 3 we will receive that $\varphi_0 = \arccos(1-\Phi/2) \simeq 1{,}378\,532\,839 \simeq 1+\phi^2 \leftrightarrow 78{,}989\,843\,166^o$,

$AD = M_1 = \sqrt{\Phi}$, $BD = M_{-1} = \sqrt{\phi}$, $CD = M_0 = 1$, $AC = \Phi$, $BC = \sqrt{1+\phi}$,

$\angle A = \mathrm{arctg}\sqrt{\phi} = 38{,}172\,707\,627^o$, $\angle B = \mathrm{arctg}(\sqrt{\Phi}) \simeq 51{,}827\,292\,373^o$,

$$W_{dhe} = \frac{(\sqrt{\phi}+1)(1+\sqrt{\Phi})}{1\cdot(\sqrt{\phi}+1+\sqrt{\Phi})} = 1 + \frac{1}{1+\sqrt{\phi}+\sqrt{\Phi}} \simeq 1{,}326\,992\,830 \neq \Phi^2/2$$

8. But this $_\triangle ABC$ is also simultaneously the Kepler's and meta-triangle (!!) as

$$c/b = (\sqrt{\phi}+\sqrt{\Phi})/\Phi = b/a = \Phi/\sqrt{1+\varphi} = \sqrt{\Phi} \simeq 1{,}272\,019\,650,$$

$$a\cdot b = \sqrt{\Phi}\cdot\Phi = c = \sqrt{\phi}+\sqrt{\Phi} \simeq 2{,}058\,171\,027$$

In the completion it must be pointed out some other interesting relations, which were received in the MP dynamics from physical reasons and which are revealed a new results in the golden ratio geometry and trigonometry. The omnipresence of the golden ratio constants ϕ, Φ is caused, in particular, by the fact that <u>any integral number</u> can be <u>precisely</u> expressed through their combinations, including the symmetric. For example, $1 = \phi\cdot\Phi$, $2 = \phi\cdot\Phi + \Phi\cdot\phi$, $3 = \phi^2 + \Phi^2$, $4 = \Phi^3 - \phi^3$, $5 = (\phi+\Phi)^2$, $7 = \phi^4 + \Phi^4$, $11 = \Phi^5 - \phi^5$ and so on.

The relation of the components of the bob impulse $\vec{p}$ on the vertical and horizontal axes $|p_y / p_x| = tg(\varphi) = \phi^2,\ \phi,\ \sqrt{\phi},\ \sqrt{\Phi},\ \Phi,\ \Phi^2$ are realized accordingly at following angles φ of the bob deviation from a vertical:

$$\varphi_1(\phi^2) = arctg(\phi^2) = arccos(\frac{\Phi}{\sqrt{\phi^2+\Phi^2}}) \simeq 20{,}905^o, \qquad \varphi_2(\phi^2) = \pi - \varphi_1(\phi^2)$$

$$\varphi_1(\phi) = arctg(\phi) = arccos\left(\frac{\Phi}{\sqrt{\Phi^2+1}}\right) \simeq 31{,}717^o, \qquad \varphi_2(\phi) = \pi - \varphi_1(\phi)$$

$$\varphi_1\left(\sqrt{\phi}\right) = arctg(\sqrt{\phi}) = arccos\left(\sqrt{\phi}\right) \simeq 38{,}173^o, \qquad \varphi_2(\sqrt{\phi}) = \pi - \varphi_1(\sqrt{\phi})$$

$$\varphi_1\left(\sqrt{\Phi}\right) = arctg(\sqrt{\Phi}) = arccos\left(\phi\right) \simeq 51{,}827^o, \qquad \varphi_2(\sqrt{\Phi}) = \pi - \varphi_1(\sqrt{\Phi})$$

$$\varphi_1\left(\Phi\right) = arctg\Phi = arccos\left(\sqrt{\frac{\phi}{\phi+\Phi}}\right) \simeq 58{,}283^o, \qquad \varphi_2(\sqrt{\Phi}) = \pi - \varphi_1(\sqrt{\Phi})$$

$$\varphi_1\left(\Phi^2\right) = arctg\Phi^2 = arccos\left(\frac{2}{\sqrt{\Phi^5-\phi^3}}\right) \simeq 58{,}283^o, \qquad \varphi_2(\Phi^2) = \pi - \varphi_1(\Phi^2)$$

It is important that the relative sum and difference of the potential energies of the bob at these angles $U_1(\varphi_1)$, $U_2(\varphi_2)$, normalized on $U_{max} = 2mgL$, are equal:

$$[U_1(\varphi_1) + U_2(\varphi_2)] / U_{max} = [(1-\cos\varphi_1) + (1-\cos\varphi_2)] / 2 = 1,$$

$$[(U_2(\varphi_2) - U_1(\varphi_1)] / U_{max} = \cos(\varphi_1) = -\cos(\varphi_2)$$

The angle of maximum deflection φ_{max} of the MP rod (or string) is equal

$$\varphi_{max}(\kappa) = arccos(1-2\kappa^2), \qquad \varphi_{max}(1) = \pi$$

At the same time the angle of deflection φ_0 at which the component of gravitational force along the MP rod (or string) is equal to centripetal force and, therefore, the tension of the rod (or string) is equal to zero is determined by the expressions:

$$\varphi_0(\kappa^2) = \arccos[2(1-2\kappa^2)/3],$$

$$\pi/2 < \varphi_0(\kappa^2) \le \varphi_0(\kappa^2 = 1) = \mathrm{arctg}[-(\phi + 1/2)] \simeq 2{,}300\,524 \leftrightarrow 131{,}810\,315^{o},$$

$$\varphi_0(\kappa^2 = 1/2 + 3\phi/4) \simeq 2{,}237\,035 \leftrightarrow 128{,}172\,707^{o},$$

$$\mathrm{tg}[\varphi_0(\kappa^2 = 1/2 + 3\phi/4)] = -\sqrt{\Phi}, \quad \cos[\varphi_0(\kappa^2 = 1/2 + 3\phi/4)] = -\phi,$$

$$\varphi_0(\kappa^2 = 1/2 + 3\sqrt{\phi}/4\sqrt{\phi+\Phi}) = \mathrm{arctg}(-\Phi) \simeq 2{,}124\,371 \leftrightarrow 121{,}717\,474^{o},$$

$$\cos[\varphi_0(\kappa^2 = 1/2 + 3\sqrt{\phi}/4\sqrt{\phi+\Phi})] = \sqrt{\phi/(\phi+\Phi)} \simeq 0{,}525\,651$$

In the case when the bob is suspended on the string, the bob, after passing the angle φ_0, moves not on a circle any more, but on a parabola as the body moving in a uniform gravitational field.

For the MP it is possible to find many other interesting relationships which are also precisely expressed through constants ϕ, Φ.

For example, if the point of suspension of the MP string be shifted with velocity v_O in the horizontal direction when the MP is on a platform in a moving object. In the reference system connected with this object, the bob will begin to move on a circle. According to the Newton`s second law and the energy conservation law for the top point of a trajectory A we will receive

$$mg + F_A = mv_A^2/2, \qquad mv_O^2/2 = 2mgL + mv_A^2$$

where F_A is the string tension at the point A. Proceeding from the condition

$$F_A = m(v_O^2 - 4gL)/L - mg \ge 0$$

we will receive that minimum speed of the movement of the suspension point O at which the bob will begin to move on a circle, is equal $v_{Omin} = 5gL = (\phi + \Phi)^2 gL$.

Similarly proceeding from the Newton's second law and the energy conservation law for the string deviation by 90^o degrees at the point B

$$F_B = mv_B^2 / L, \qquad mv_A^2 / 2 = mv_B^2 / 2 + mgL,$$

we will receive that the string (or rod) tension in the horizontal position F_B is also symmetrically and beautifully expressed through the constants of the golden ratio: $F_B = 3mg = (\phi^2 + \Phi^2)mg$.

Thus, by means of the MP it is possible to register not only the accelerations, but also the speeds of movement.

At last, we will return to fig. 4. $\triangle ABC$ in variant 3 is a half of the frontal section of pyramid of Cheops [11,12] - $\wedge ABA'$, which is formed at a mirror reflection of $\triangle ABC$ from the side AB. Therefore, if to use the scale in which a unit of length is CD=h=1 then all ratios received in variant 3 will be satisfied also for the pyramid of Cheops, i.e. a half of its frontal section will be simultaneously both the Kepler's and the meta-triangle. In other scales when $CD = h \neq 1$, the condition of existence of the meta-triangle $a \cdot b = c$ is not satisfied.

According to [11,12] for the pyramid of Cheops the condition of receiving the ideal wurf is satisfied for 3 pieces of the $\wedge ABA'$ – the height of pyramid $H = BC = \Phi$, the radiuses of the circle $r_S = \phi$ and the radius of the semi-circle $r_{SS} = CD = h = 1$, entered in the $\wedge ABA'$: $W_{r_S r_{SS} H} = \Phi^2 / 2$. At the same time $H = r_S + r_{SS}$

Let's emphasize that the very important criterion of the nonrandomness of existence of the specified regularities for the Pyramid of Cheops are, firstly, the fact that exactly at $H = \Phi$ the function $\Delta r(H) = r_{SS}(H) - r_S(H)$ has the extremum (maximum), and, secondly, the function $\Sigma r(H) - H = r_{SS}(H) + r_S(H) - H$ changes a sign and passes through zero.

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Foucault pendulum in the Parisian Pantheon